

Black hole microstates and attractor without supersymmetry

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ABSTRACT: Due to the attractor mechanism, the entropy of an extremal black hole does not vary continuously as we vary the asymptotic values of various moduli fields. Using this fact we argue that the entropy of an extremal black hole in string theory, calculated for a range of values of the asymptotic moduli for which the microscopic theory is strongly coupled, should match the statistical entropy of the same system calculated for a range of values of the asymptotic moduli for which the microscopic theory is weakly coupled. This argument does not rely on supersymmetry and applies equally well to nonsupersymmetric extremal black holes. We discuss several examples which support this argument and also several caveats which could invalidate this argument.

KEYWORDS: Black Holes in String Theory, Superstrings and Heterotic Strings.

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1. Introduction

One of the important successes of string theory is that one can obtain a statistical understanding of the thermodynamic Bekenstein-Hawking entropy of certain supersymmetric black holes in terms of microscopic counting [1]. The main theoretical tool in much of this work is the BPS property of these supersymmetric black holes. A BPS state in theories with $\mathcal{N} = 2$ or more supersymmetry belongs to a short representation of the supersymmetry algebra. As a result, under suitable conditions, the number of BPS states cannot jump discontinuously under smooth variations of the coupling constant and other moduli. The spectrum of BPS states with a given assignment of charges can then be reliably computed at weak coupling and then analytically continued to the strong coupling regime where the same state is described by a supersymmetric black hole. This allows us to compare the statistical entropy with the Bekenstein-Hawking entropy even though they are calculated in different regions in the coupling constant space. In addition the BPS property of these states leads to considerable computational simplification. Exact solutions describing the

corresponding black hole in supergravity can be found by solving first-order Killing spinor equations instead of second-order equations of motion.

A further significant simplification results from the ‘attractor mechanism’ noted first in the context of supergravity [2–4] and generalized to theories with higher derivative terms in [5–8]. The moduli fields in a black hole background vary radially and get attracted to certain specific values at the horizon which depend only on the quantized charges of the black hole under consideration. As a result, the macroscopic entropy is determined purely in terms of charges and is independent of the asymptotic values of the moduli. This is consistent with the fact that the microscopic entropy is also independent of the asymptotic moduli due to the BPS property of the state it counts. The attractor values of the moduli are determined by solving a set of ‘attractor equations’ which are purely *algebraic*. Thus, with the attractor mechanism, the problem of finding the entropy of supersymmetric black holes is simplified enormously and reduced to solving algebraic equations instead of non-linear second or higher order differential equations.

Using the generalized attractor mechanism, and using the proposal for mixed statistical ensemble proposed in [9], it has recently become possible to carry out a far more detailed comparison between microscopic and macroscopic entropy. For a number of examples of both small and large black holes the two entropies agree to all orders in a perturbation theory in inverse charges going well beyond the thermodynamic Bekenstein-Hawking result [10–15].

Much of this success is crucially tied to supersymmetry and it is interesting to ask if some generalization to non-supersymmetric black holes is possible. Indeed, there are already a number of indications that the attractor mechanism as well as the agreement between thermodynamic and statistical entropy could work even without supersymmetry for extremal black holes.

- For many extremal but non-supersymmetric black holes within string theory, both in four and five dimensions, the macroscopic entropy agrees with the microscopic degeneracy of states computed at weak coupling [16–19]. Such an agreement is *a priori* quite mysterious, because these black holes are not ‘nearly supersymmetric’ in any sense and break supersymmetry completely. Since they belong to a long representation of the supersymmetry algebra, one cannot invoke the argument given above for the analytical continuation of their spectrum from weak coupling to strong coupling in an obvious way.
- The attractor mechanism is a consequence not so much of supersymmetry but rather of the near horizon extremal geometry which is $AdS_2 \times S^n$ in $n + 2$ dimensions. For a general class of two derivative actions describing gravity coupled to scalar fields and abelian gauge fields, extremal black holes are known to exhibit attractor phenomenon under certain conditions even without supersymmetry [20–31]. For the non-supersymmetric black holes mentioned above, where exact supergravity solutions are known, several moduli do get attracted to fixed values at the horizon irrespective of their values at asymptotic infinity.

- For a completely general class of gravity actions including arbitrary higher derivative interactions, assuming an extremal near horizon geometry but without assuming supersymmetry, the attractor values of moduli can be obtained by extremizing an ‘entropy function’. The value of the function at the extremum gives the full Bekenstein-Hawking entropy of these black holes [21, 32] after inclusion of higher derivative corrections to the entropy formula following Wald’s procedure [33–36].

Motivated by these results, we investigate the question of the microscopic interpretation of the entropy of non-supersymmetric but extremal black holes within string theory. In §2 we propose an argument as to why the microscopic and macroscopic entropy of an extremal black hole should agree despite the fact that they are calculated in different regimes in the coupling constant space. Our argument does not rely on supersymmetry but relies rather on the attractor phenomenon. The basic underlying idea is the following.

While in absence of supersymmetry we lack an argument that allows us to continue the expression for the statistical entropy from the weak coupling to the strong coupling regime, the attractor mechanism, – which tells us that the entropy of an extremal black hole does not change as we vary the asymptotic values of the moduli fields, – allows us to continue the expression for the black hole entropy from the strong coupling to the weak coupling regime where it can be compared with the statistical entropy. Based on this argument we present a conjecture that for all extremal black holes, the macroscopic entropy will agree with the weak-coupling microscopic entropy as long as certain conditions are satisfied. In particular the geometry must approach $AdS_2 \times S^n$ form near the horizon which can be modified but not destabilized by higher derivative corrections, an interpolating solution must exist that connects the weakly coupled region at asymptotic infinity to the attractor geometry near the horizon and the near horizon field configuration should not jump discontinuously under a continuous variation of the asymptotic moduli from strong to weak coupling regime. Besides providing new information on extremal non-supersymmetric black holes, our argument also provides a new explanation of why the statistical and black hole entropy agree for extremal BPS black holes.

One subtlety that arises in the comparison between statistical entropy and the black hole entropy involves the precise definition of the statistical entropy of extremal black hole. Since the mass of a non-BPS state can change continuously as a function of the coupling, the degeneracy of strictly lowest energy states in a given charge sector may change as we vary the coupling. A more appropriate definition would be the logarithm of the total number of states within a given range of the lowest energy eigenvalue, or equivalently the statistical entropy calculated for a small but non-zero temperature. On the other hand there is also a potential problem in defining the entropy of a strictly extremal non-BPS black hole due to the fact that some of the flat directions of the leading entropy function could be lifted due to higher derivative terms in the action, and the full entropy function may not have a non-trivial extremum. In this case we shall have a runaway behavior of the moduli fields as we approach the horizon of an extremal black hole, and we need to control this by introducing a small amount of non-extremality. Both these issues as well as their relationship are discussed in § 3.

Once we introduce a small amount of nonextremality, the entropy is no longer strictly independent of the asymptotic moduli. Thus the validity of our argument will depend on the extent to which the entropy begins to depend on the ‘flat directions’. We need to analyze the dynamics on a case by case basis to settle this issue. Often it is possible, based on other arguments, to determine the order at which the equations of motion associated with the flat directions begin receiving a non-trivial contribution. The entropy function formalism then tells us that the dependence of the entropy on the flat directions also begins at that order, – the point being that the function whose extremization gives the equations of motion is the same function whose value at the extremum gives the entropy. If the order at which this dependence begins remains subleading as we vary the moduli from the strong coupling to the weak coupling regime, then our argument about the equality of microscopic and macroscopic entropy remains valid. An example of such a situation can be found in § 6.

In § 4 we review various known examples of non-supersymmetric extremal black hole solutions [18, 16, 24] where the microscopic entropy is known to agree with the macroscopic entropy despite lack of supersymmetry. In § 5 we show in detail how our argument works for a specific example of five dimensional black hole described in § 4. In § 6 we explore, with the help of some examples, what happens if some of the marginal directions of the entropy function get lifted after inclusion of higher derivative terms and the resulting entropy function does not have an extremum. In this case for a strictly extremal black hole the geometry and other background fields keep evolving as we go down the infinite throat of the would be AdS_2 and we never reach the near horizon $AdS_2 \times S^n$ geometry. However we show that by introducing a small amount of non extremality we can tame this runaway behavior and get a black hole solution sufficiently close to the original extremal black hole solution in the absence of higher derivative terms. Thus the entropy function method can be used to calculate the entropy of such black holes. Furthermore our argument showing the independence of the entropy of the asymptotic moduli will hold for the entropy of such black holes to a good approximation.

Most of the analysis in this paper is based on the assumption that the near horizon geometry of the black hole has an AdS_2 factor. In many examples in string theory black holes one finds that this AdS_2 factor combines with an internal compact circle to produce a locally AdS_3 space. In such cases the additional symmetries of AdS_3 allows us to derive results which are much more powerful than the ones based on the assumption of only the AdS_2 geometry [37–39]. In section § 7 we review the results obtained using the assumption of AdS_3 near horizon geometry, and also discuss the relative strength and weakness of this approach compared to the AdS_2 based approach. In section 8 we generalize our analysis to include the case of extremal rotating black holes.

2. Microstate counting and the non-supersymmetric attractor

In this section we shall argue that subject to certain conditions being satisfied, the microscopic entropy of an extremal black hole must match the macroscopic entropy even in the absence of supersymmetry. The issue at hand is the following. Let us take all the non-zero charge quanta to be large and (say) of the same order N and let λ be the closed

string coupling constant. The microscopic entropy of this system can be calculated in the range of λ where we can describe the dynamics of the system in terms of a set of weakly interacting degrees of freedom. Typically this requires a combination involving positive powers of λ and N to be small, e.g. for a D-brane system this requires the 't Hooft coupling λN to be small. We shall call this the weak coupling region of the moduli space. But in this region gravity is weak and the horizon of a would-be black hole carrying a fixed set of charges form at such a small radius that the classical supergravity description breaks down at the horizon. Hence there is no conventional black hole solution describing the system. If we now keep the charge quanta fixed but increase the coupling constant, then the horizon radius of the would be black hole would grow and eventually we get a regular black hole solution. In this region we can reliably calculate the black hole entropy, but the microscopic degrees of freedom become strongly interacting and hence we cannot reliably compute the microscopic entropy. We shall call this the ‘strong’ coupling region.¹ The question is: How can we compare the two entropies calculated in two different regions in the coupling constant space?

For supersymmetric states the BPS condition allows us to analytically continue the expression for the statistical entropy computed for weak coupling into the regime of ‘strong’ coupling. This analytic continuation is justified by the classic argument of Witten and Olive [40] that relies on the fact that a BPS state belongs to a short representation of the supersymmetry algebra and hence the number of BPS states cannot jump discontinuously as we continuously vary the parameters of the theory. Thus if one had a similar argument for the non-renormalization of the degeneracy of states for the non-BPS states, then we could continue the answer for the statistical entropy from weak coupling region to ‘strong’ coupling region, and compare this with the black hole entropy. Unfortunately such a non-renormalization theorem is not available for the statistical entropy of non-BPS states.

This is where the attractor mechanism comes to our rescue. This allows us to run the argument backwards, — namely we calculate the black hole entropy in the ‘strong’ coupling region, and then continue the result to the weak coupling region using the fact that the black hole entropy is independent of the asymptotic value of the string coupling constant λ . In the weak coupling region we can compare the result with the statistical entropy.

Let us elaborate on this point in some detail. We can view the black hole geometry as an interpolating geometry from the asymptotic infinity to the horizon. At large coupling the curvatures are small everywhere in the geometry. Thus we can calculate the entropy of the black hole as a systematic expansion in inverse powers of N using Wald’s formula or equivalently the entropy function defined in [21, 32]. For small coupling, as we move radially inwards, the spacetime will typically develop regions of high curvatures. In these regions, it would be necessary to go beyond the supergravity approximation and include the higher derivative corrections to the low energy effective action. We can formally include all higher derivative corrections keeping all terms in the effective action. Then *assuming* that

¹For large N this can be done by keeping λ small so that the asymptotic theory is still weakly coupled. Thus by ‘strong’ coupling region we shall mean that the microscopic degrees of freedom of the black hole are strongly coupled but the asymptotic theory is weakly coupled.

the fully corrected spacetime geometry exits into an $AdS_2 \times S^n$ geometry (possibly with large curvature or large coupling constant) as we move radially inwards, one can formally compute the full Wald entropy using the entropy function that incorporates the effects of the higher derivative terms. The parameters labeling the near horizon field configuration are obtained by extremizing the entropy function with respect to these parameters, and the entropy is given by the value of the entropy function at this extremum. If the entropy function has a unique extremum, then of course the near horizon field configuration and the entropy are uniquely determined by the entropy function and cannot depend on the asymptotic moduli. If the entropy function has one or more flat directions then not all the moduli at the horizon are determined in terms of the charges and could depend on the asymptotic values of the moduli fields. However the entropy, being the value of the entropy function at the extremum, will not depend on the asymptotic moduli [21, 32]. Thus the final entropy will have the same value for ‘strong’ and weak coupling and the entropy will continue to have the same perturbative expansion in inverse powers of N where N stands for some typical charge of the black hole.

We can present the argument in another way that does not directly refer to having a near horizon $AdS_2 \times S^n$ geometry at weak coupling. Let us denote by $f(\lambda)$ the black hole entropy as a function of λ . Now the analysis based on the entropy function tells us that for large λ it is strictly independent of λ provided the contribution of the higher derivative terms do not destabilize the $AdS_2 \times S^n$ near horizon geometry. If we now assume further that $f(\lambda)$ is an analytic function of λ , then it must be strictly independent of λ in the full complex λ plane, or a region in the complex λ plane containing the ‘strong’ coupling region in which $f(\lambda)$ is analytic. If this region includes the weak coupling region then $f(\lambda)$ in the weak coupling region will have the same value as in the ‘strong’ coupling region.

At this point special mention must be given to small black holes – black holes which describe elementary string excitations. In this case there is no regular horizon in the supergravity approximation; the closest analog to the attractor geometry is a scaling region where the solution becomes independent of all asymptotic parameters [41, 42]. Analyzing the behavior of the solution in this scaling region and knowing certain general structure of the string effective action one can show that up to an overall normalization factor that is not determined by the scaling argument, the entropy of the small black hole agrees with the statistical entropy computed from the elementary string spectrum [41–43]. Further analysis based on certain non-renormalization theorem then shows that the overall normalization constant also agrees [10, 38]. Given that the supergravity solution does not have a regular horizon one might wonder about the relevance of our argument in the context of small black holes. To this end we note that in order that the solution enters the scaling region we need to adjust the asymptotic coupling so that we are in the ‘strong’ coupling region in the sense described above. Otherwise before we enter the scaling region the curvature and other field strengths become strong. We then need to invoke the independence of the entropy on the asymptotic parameters to argue that the black hole entropy remains the same as we go to the weak coupling region.

These arguments are predicated on several important assumptions which we list below:

1. We assume that after including the higher derivative corrections, the near horizon geometry still is of the form $AdS_2 \times S^n$ so that we can apply the formalism of [21, 32]. Note that it does not require a detailed knowledge of the interpolating geometry, and not even the complete details of the near horizon field configuration but only that it exits into a near horizon attractor geometry of the form $AdS_2 \times S^n$. Experience with small and large black holes indicates that this assumption is likely to be satisfied at least in a large number of cases. In fact in the case of small black holes the higher derivative corrections actually create the $AdS_2 \times S^n$ near horizon geometry even though in the supergravity approximation the geometry is singular [10, 11]. In general it is quite difficult to analyze the details of the full geometry reliably once the curvatures are large unless there is some help from supersymmetry. However, the entropy in many examples appears to be more robust than other inessential details of the geometry.

The arguments based on analyticity bypasses the need of having $AdS_2 \times S^n$ near horizon geometry in the weak coupling region, but it requires existence of $AdS_2 \times S^n$ geometry in the ‘strong’ coupling region even after inclusion of all the higher derivative corrections. As we shall discuss, this may not always be true if some of the flat directions of the entropy function are lifted after inclusion of higher derivative terms and the resulting entropy function has no extremum.

2. A key ingredient in our argument is the fact that for an $AdS_2 \times S^n$ near horizon geometry the entropy does not change as we continuously vary the asymptotic coupling constant. This in turn follows from the fact that the black hole entropy is obtained by extremizing an entropy function with respect to the parameters labeling the near horizon geometry. For a local action there is a well defined algorithm for constructing the entropy function from the local Lagrangian density [21, 32]. But typically fully quantum corrected effective action has non-local terms and it is not *a priori* guaranteed that the notion of entropy function will continue to hold in the presence of such terms. In our argument we have implicitly assumed that the entropy function formalism continues to hold for full quantum corrected effective action which could in principle contain non-local terms as well. This assumption is essential in cases where quantum corrections to the effective action are important in the near horizon geometry of the black hole.²
3. Even if both the above assumptions are correct, a discontinuous change in the entropy may arise as we vary λ if λ crosses over to a different basin of attraction. Typically this will move the near horizon geometry to a different extremum of the entropy function and will change the value of the entropy. Clearly our argument will break down if this happens.

²Even if the string coupling is small at the horizon, some other parameters, e.g. inverse sizes of the compactification manifold, may become large, forcing us to use a dual description. In this dual description the string coupling may not be small.

4. Typically in the supergravity approximation the entropy function has several flat directions both for BPS and non-BPS extremal black holes. Once higher derivative corrections are taken into account some of these flat directions may be lifted. Generically for supersymmetric black holes there are non-renormalization theorems which prevent this, but there is no such result for non-supersymmetric black holes. If the resulting entropy function has an extremum where the curvatures and other field strengths are small we can still calculate the entropy function in a systematic expansion in inverse powers of N , and higher derivative terms will give rise to small corrections to the leading entropy. However there could be potential problem if the resulting entropy function has no extremum. In this case if we follow the radial evolution of various fields, there will be a runaway behavior as we approach the horizon and we shall not get an $AdS_2 \times S^n$ near horizon geometry. Even if there is an extremum but at the extremum the near horizon geometry has large curvature where the higher derivative corrections are important, then there can be large correction to the leading order result for the entropy. As a result even at ‘strong’ coupling, when the curvature is small everywhere in the supergravity approximation, higher derivative corrections will modify the solution in a non-trivial way that would seem to completely invalidate the leading order result.

One way to avoid this problem is to consider slightly non-extremal black holes instead of exactly extremal black holes. In this case the near horizon geometry is no longer $AdS_2 \times S^n$, but for sufficiently large charges and small extremality parameter there will be a long throat region where the geometry will be approximately $AdS_2 \times S^n$. We can then calculate the approximate value of the entropy by evaluating the entropy function in this region. For our argument to be valid, we need to assume that the entropy of such a black hole remains approximately independent of the asymptotic values of the moduli fields all the way from the strong coupling to the weak coupling region. This issue together with its microscopic counterpart will be discussed in more detail in § 3, and will be illustrated with example in § 6.

5. Another important assumption that has gone into our argument is the identification of the extremal black hole with the lowest mass state for a given set of charges. As explained above, an extremal black hole is defined by the requirement that its near horizon geometry is $AdS_2 \times S^n$. The entropy function formalism allows us to compute the entropy of these black holes for a given set of charges but does not give us any information about its mass. On the other hand when we compute the degeneracy of states by identifying the black hole with a configuration of branes in string theory, we typically calculate the degeneracy of states with the lowest mass consistent with a given set of charges. In our argument we have implicitly assumed that these two requirements are identical, i.e. an extremal black hole always describes the lowest mass state with a given set of charges. This is of course true when the space-time curvature is small everywhere outside the black hole horizon, but may break down when there are regions of strong curvature in the black hole solution.

6. A related issue is that of a precise definition of statistical entropy of an extremal black hole. In the case of supersymmetric black holes there is a clear distinction between BPS states and nearly BPS states since they belong to different representations of the supersymmetry algebra. Thus we can define the statistical entropy of BPS states by counting the number of BPS supermultiplets. But in absence of supersymmetry there is no such clear distinction between the lowest mass states and other states and it would seem more natural to define the statistical entropy as the logarithm of the total number of states with mass within a small range of that of the lowest mass state. We shall discuss this point in more detail in § 3. For the time being we note that this fits in well with the requirement of introducing a small amount of non-extremality in the black hole description due to lifting of the flat directions since the latter corresponds to introducing a small temperature or equivalently defining the entropy by counting the total number of states within a small energy range around the lowest energy state.
7. In our analysis we have assumed that the black hole under consideration is stable. For BPS black holes this follows as a consequence of supersymmetry, but this need not be true for non-BPS black holes. Nevertheless we expect that as long as the black hole does not have any classical instability, it should at least be long lived (if not stable) since there is no Hawking radiation from extremal black holes and we should be able to define the entropy of such black holes. On the microscopic side the corresponding microstates should also be long lived since they are the lowest mass single particle states for a given charge, and hence the phase space available to them for decaying into lower mass particles should be small. Hence it should be possible to define the entropy on both sides and carry out the comparison of the black hole entropy with the statistical entropy. Notwithstanding these general arguments, stability of extremal non-supersymmetric black holes clearly is an issue that should be examined in detail on a case by case basis. Our arguments will apply only to the cases where the black hole is stable or long lived.

Subject to these caveats, our arguments suggest the following conjecture.

Conjecture: *Thermodynamic entropy of extremal black holes in string theory matches with the statistical entropy determined by counting of underlying microstates at weak coupling.*

This conjecture says that the attractor mechanism in effect provides a non-renormalization theorem for the degeneracy of states which carry the lowest mass for given charge.

In § 4 and § 5 we will elaborate on this argument through various examples.

3. Defining the entropy of non-BPS extremal black holes

If our conjecture is correct in full generality, then the reasons for the agreement between macroscopic and microscopic entropy appear to go well beyond the usual arguments from BPS stability. In this section we address the question of precise definition of the microscopic and macroscopic entropy that goes into the aforementioned correspondence.

First let us consider the case of BPS black holes. Our conjecture implies that the macroscopic entropy of an extremal black hole should agree with the weak-coupling statistical entropy. By definition, statistical entropy is always the logarithm of the absolute number of microstates carrying a given set of macroscopic charges. However, often in comparing the statistical and black hole entropy for BPS states one uses an index rather than the absolute number to compute the statistical entropy. The rationale behind this is the underlying assumption that in general at ‘strong’ coupling whatever states could combine with other states to become non-BPS will do so, and only the index worth of states will remain in the spectrum of BPS states. Thus at ‘strong’ coupling the absolute number of microstates is equal to the index. There are some notable exceptions to this rule; the simplest examples being the ones discussed by Vafa in [44] for supersymmetric black holes. In many cases discussed there the absolute number of black hole microstates with three charges scales as $N^{3/2}$ in agreement with the entropy whereas the index scales as N . Except for this ambiguity, the statistical entropy of a BPS black hole is well defined, since a BPS state can be clearly distinguished from a non-BPS state by its supersymmetry transformation property.

The definition of macroscopic entropy of a BPS black hole is also reasonably clean. The attractor phenomenon tells us that the black hole entropy does not vary continuously as we vary the asymptotic moduli. In particular if the near horizon values of some moduli are not determined by the attractor equations, then the entropy is independent of these moduli. We also expect that in many (if not all) cases supersymmetry will prevent lifting of these flat directions by higher derivative terms and associated runaway behavior, especially if the near horizon geometry has enhanced supersymmetry as in [5, 6, 45, 8]. Hence the entropy of such black holes remains well-defined.

For non-BPS black holes the situation is much more murky. First of all, on the microscopic side there is no analog of an index, and there is no clear distinction between the lowest energy state and nearby states with slightly higher energy. Even if the lowest energy state is degenerate at zero coupling, once a small coupling is switched on the degeneracy may be lifted unless it is protected by some symmetry. This suggests that a more appropriate quantity will be the total number of states which are within a small but fixed mass range ϵ or equivalently the entropy calculated at a small but non-zero temperature. For small enough coupling when the correction to the mass of a state is smaller than the parameter ϵ , the statistical entropy calculated at zero coupling can be expected to be equal to that calculated at weak coupling. The entropy defined this way however acquires a subleading piece that depends on the precise nature of the energy cut-off as well as on the various moduli characterizing the vacuum.

Apparently independent of these considerations, the possible runaway behavior at the horizon, associated with lifting of the flat directions of the entropy function by the higher derivative corrections, may require us to introduce a slight amount of non-extremality on the black hole side. To see how it works, let us denote by ϵ the non-extremality parameter. The effect of the non-extremality parameter is to truncate the infinite throat of AdS_2 into a finite size, and as a result the near horizon geometry is no longer $AdS_2 \times S^n$. Since the original runaway behavior came from radial evolution along the infinite throat of the AdS_2 geometry, we expect that for any finite ϵ various fields will approach finite values at

the horizon instead of showing runaway behavior. However for sufficiently large charges and small ϵ there will be a region in the black hole space-time where the geometry is approximately $AdS_2 \times S^n$, and we can apply the entropy function formalism to calculate the entropy in this region. (This will be demonstrated in § 6 with the help of some examples.) Although this does not give the exact entropy which requires us to evaluate the appropriate Wald's integral at the horizon, the entropy calculated by regarding the long throat region as the near horizon geometry will continue to give an approximate value of the entropy. However the entropy calculated this way acquires a mild dependence on the asymptotic moduli since the near horizon values of the originally flat moduli depends on the asymptotic data, and the entropy function now has a piece $\Delta\mathcal{E}$ that depends on these 'flat directions'.

Even though we have presented the problems with runaway behavior at the horizon and that of defining statistical entropy of microscopic states as two separate problems, we expect them to be related. In the spirit of AdS/CFT correspondence we could identify the radial evolution of various moduli fields in the black hole description with the renormalization group (RG) evolution of various parameters in the microscopic theory describing the black hole. Thus a runaway behavior of the moduli fields in the gravity description will correspond to a runaway behavior of the parameters of the microscopic theory in the far infrared. Even if there is a non-trivial infrared fixed point where the parameters reach a finite value, either the gravity description, or the microscopic description (or both) must be strongly coupled in this region since we cannot have a configuration where both the gravity and the microscopic description are simultaneously weakly coupled. The way this problem is avoided in the case of supersymmetric black holes is by having one or more flat directions of the near horizon geometry which we can tune to go from weakly coupled microscopic description to weakly coupled gravity description. Since for non-supersymmetric black holes we expect the flat directions to be lifted in general, the only way we can avoid this problem is by introducing a small amount of non-extremality. On the black hole side it effectively cuts off the evolution of the moduli fields at certain radius. Its counterpart on the microscopic side is to introduce certain infrared cut-off. This is precisely the effect of introducing a small temperature into the system. The long throat region with approximately $AdS_2 \times S^n$ geometry on the black hole side should correspond, on the microscopic side, to a range of scale where all the β -functions are small and we have an approximately conformal quantum mechanics.

This by itself of course does not solve the problem, since again if the parameters in this throat region are such that the microscopic theory is weakly coupled, then the gravity description has strong curvature and vice versa. However often in this case we have one or more approximate flat directions which we can adjust to go from weakly coupled microscopic description to weakly coupled supergravity description. Since the entropy function does not change appreciably as we move along these flat directions we get a relation between the statistical entropy and black hole entropy. However since we now only have approximately flat directions, both entropies acquire mild dependence on the energy cut-off and asymptotic moduli in their subleading piece. As a result the comparison between the weak coupling statistical entropy and the strong coupling black hole entropy cannot be carried out to an arbitrary accuracy, but only up to terms of a certain order

which are not affected by the ambiguities in the definition of the entropy introduced due to the need of considering slightly non-extremal black holes. Clearly, the relationship is most robust for the leading term for which all the ambiguities mentioned above disappear.

There is however a potential danger with this argument. We have seen that the black hole entropy function now acquires a piece $\Delta\mathcal{E}$ which gives subleading contribution of the entropy that depends on the original ‘flat directions’. These contributions are subleading as long as the effect of lifting of the flat direction is a small effect. However in order to carry on our argument we need to vary the asymptotic moduli all the way to the weak coupling regime and this could push the near horizon field configuration to a regime where the $\Delta\mathcal{E}$ piece becomes large. If this happens then we can no longer use our argument to show the equality of macroscopic and microscopic entropy.³ The entropy function formalism by itself cannot tell us if this happens or not; we need to analyze the dynamics on a case by case basis to settle this issue. The point however is that often it is possible, based on other arguments, to determine the order at which the equations of motion associated with the flat directions begin receiving a non-trivial contribution. The entropy function formalism then tells us that the dependence of the entropy on the flat directions also begins at that order, — the point being that the function whose extremization gives the equations of motion is the same function whose value at the extremum gives the entropy. If the order at which this dependence begins remains subleading even in the transition region between weak and ‘strong’ coupling, then our argument about the equality of microscopic and macroscopic entropy remains valid. We shall illustrate this in an explicit example in § 6.

There is one class of examples discussed in this paper which require a slightly different treatment. These are the cases of small black holes. In this case the microscopic theory is that of elementary strings, and for weak coupling when we work within single string Hilbert space, this theory is free (if we work in flat space) or described by a sector of a 1+1 dimensional conformal field theory. This makes the computation of the microscopic entropy easy. As a consequence we should expect that the near horizon geometry of the corresponding black hole cannot be described within supergravity approximation. This is indeed true since the curvature at the horizon is of the order of the string scale, and there is no flat direction which we can adjust to change this. Nevertheless by varying the asymptotic parameters (on which the entropy does not depend as a consequence of the attractor mechanism) we can bring the solution to a form where certain scaling arguments apply; and we can use them to determine the dependence of the black hole entropy on the charges up to an overall numerical constant even though the horizon geometry has strong curvature [41–43]. In the case of four dimensional black holes it has been possible to even compute the overall numerical constant using various additional techniques [10, 11, 38, 37]. However since these computations require us to go beyond supergravity approximation, there is no obvious contradiction with the fact that the microscopic theory is weakly coupled.

³In fact in the weak coupling regime the statistical entropy of states within the mass range ϵ as discussed above is equal to that computed in the free theory, and hence is independent of the coupling constant. So the issue really is whether the entropy acquires a non-trivial dependence on the coupling constant in the transition region between the weak and the ‘strong’ coupling.

4. Extremal black holes without supersymmetry

In this section we will give several examples of extremal black holes for which the weak coupling value of the statistical entropy agrees with the ‘strong’ coupling value of the black hole entropy. We first discuss two simple examples in § 4.1 and § 4.2 and then turn to more general black holes in type-II and M-theory on Calabi-Yau spaces in § 4.3.

4.1 A Nonsupersymmetric black hole in five dimensions

Let us consider heterotic string theory compactified on $K \times S^1$ where K is either T^4 or $K3$, resulting in a theory with sixteen or eight supersymmetries. We denote by x^m for $m = 6, 7, 8, 9$ the coordinates along K , and by x^5 the coordinate of the S^1 .

A basic example of a non-supersymmetric state in this theory is the following. Consider a fundamental heterotic string winding state wrapping w times along S^1 and carrying quantized momentum n along the same direction. Such a state satisfies the Virasoro constraint $N_L - N_R = 1 + nw$, where N_L is the oscillator number of the left-movers and N_R is the oscillator number of the right movers.⁴ When $n > 0$ and large, this constraint is satisfied by states which are in the right-moving ground state but carry arbitrary left-moving oscillation. Since the supersymmetries are carried by the right movers, this state, which we refer to as the F1-P state, is BPS [46]. On the other hand, when $n < 0$ and large, this constraint is satisfied by a state in the left-moving ground state but carrying arbitrary right-moving oscillations. Such a state, which we refer to as the F1- \bar{P} state is no longer supersymmetric, and indeed breaks supersymmetry completely. The F1-P state corresponds to a supersymmetric small black hole and the F1- \bar{P} state corresponds to a non-supersymmetric small black hole. We thus see that we can go from a supersymmetric state to a non-supersymmetric state simply by flipping the sign of the momentum. This is a consequence of the fact that the 1+1 dimensional world-sheet theory of the heterotic winding string is chiral and only the right-movers carry supersymmetry.

In the type-I description, the heterotic fundamental string is dual to the solitonic D1 brane [47–49] which is also chiral. Because of the chirality, the direction of the momentum along the soliton determines whether the solution is supersymmetric or not. The D1-P state is supersymmetric and the D1- \bar{P} state is non-supersymmetric.

So far we have considered states which correspond to small black holes, i.e. black holes which have vanishing entropy in the supergravity approximation. To get a state that corresponds to a large black hole with finite area in supergravity, we add D5 branes wrapped on $K \times S^1$ and consider D1-D5-P or D1-D5- \bar{P} state. Let us denote the D1 and D5-brane charges and momentum along S^1 by Q_1 , Q_5 and n respectively. Here Q_1 and Q_5 are positive and n can be positive or negative. Since for $n < 0$ supersymmetry is broken completely before adding the D5 branes, it continues to be broken even after adding the D5 branes. The counting of states for this configuration in the perturbative regime, where the ’t Hooft coupling of the gauge theory describing the low energy dynamics of the brane system is small, can be performed as in [18]. The dominant contribution to the entropy

⁴In our convention the left-movers carry positive momentum along S^1 . This differs from the convention of several other papers in the literature where left-movers carry negative momentum along S^1 .

comes from the 1-5 strings, localized on the effective 1-brane along the x^5 coordinate. There are $4Q_1Q_5$ bosons and as many Majorana fermions coming from the bi-fundamentals from the 1-5 sector along this effective brane. Thus the left as well as right-moving central charge of the CFT describing the dynamics of the effective string is $6Q_1Q_5$ and Cardy formula gives the resulting entropy to be $2\pi\sqrt{Q_1Q_5|n|}$ for both signs of n . On the other hand the black hole solution describing this configuration is also easy to construct in the supergravity approximation. One just takes the black hole solution describing the D1-D5-P system or D1-D5- \bar{P} system in the type IIB string theory [1] (both of which are supersymmetric) and interprets it as a black hole solution in the type I theory after the orientifold projection. From this it is clear that the black hole will have the same entropy for either sign of n ; indeed the part of the low energy effective action of the type I string theory that is relevant for describing this black hole solution has a Z_2 symmetry (that it inherits from the parent type IIB theory and is broken once we take into account the effect of the orientifold plane and the D9-branes) that allows us to relate the black hole solutions for n and $-n$. The answer for the black hole entropy in the supergravity approximation is $2\pi\sqrt{Q_1Q_5|n|}$.

Thus we see that the statistical entropy based on weak coupling counting agrees with the entropy of the corresponding black hole which forms only when the 't Hooft coupling is large. We thus have an agreement between the macroscopic and microscopic entropy even though the states under consideration for $n < 0$ break supersymmetry completely and maximally.

4.2 A nonsupersymmetric black hole in four dimensions

In this section we consider heterotic string theory compactified on $K \times S^1 \times \tilde{S}^1$ where again K is either T^4 or $K3$, resulting in a theory with sixteen or eight supersymmetries. We denote by x^m for $m = 6, 7, 8, 9$ the coordinates along K , and by x^5 and x^4 the coordinates of the S^1 and \tilde{S}^1 respectively.

To obtain a four-charge large black hole in four dimensions we add Kaluza-Klein 5-branes extending along 56789 directions to the configuration described in § 4.1. Since the type I D5-brane corresponds to heterotic 5-brane lying along the 5-6-7-8-9 direction, in the heterotic description we have a configuration F1-NS5-KK5-P or F1-NS5-KK5- \bar{P} . Let x^μ be the coordinates of the noncompact four dimensional spacetime in which the black hole is located. The relevant vector potentials for describing the black hole solution are $G_{4\mu}$ and $G_{5\mu}$ coming from the metric and $B_{4\mu}$ and $B_{5\mu}$ coming from the 2-form. The F1 and P (or \bar{P}) are electrically charged and couple to $G_{5\mu}$ and $B_{5\mu}$ respectively. KK5 and NS5 are magnetically charged and couple to $G_{4\mu}$ and $B_{4\mu}$ respectively. We always label the states in this heterotic description and denote by Q_1 , Q_5 , \tilde{Q}_5 , and n the numbers of F1-strings, NS5-branes, KK5-branes, and momentum in this duality frame.

The weak coupling counting is done most easily in the type-I' description as in [16]. First using the heterotic - type I duality we map this system to a D1-D5-KK5-P or D1-D5-KK5- \bar{P} state in type-I theory. If we now T-dualize the type-I theory along the x^4 direction, then we obtain a D2-D6-NS5-P or D2-D6-NS5- \bar{P} state in type-I'. The resulting configuration has Q_5 D6-branes wrapping 456789 directions, Q_1 D2-branes wrapping 45 directions, \tilde{Q}_5 NS5-branes wrapping 56789 directions and momentum n flowing along the

5 direction. In addition there are O8-planes and D8-branes at the two ends of the 4 direction. Since D2-branes can end on an NS5-brane [50], the presence of \tilde{Q}_5 NS5-branes give rise to effectively $Q_1\tilde{Q}_5$ D2-branes. Therefore the microscopic entropy is given by $2\pi\sqrt{Q_1Q_5\tilde{Q}_5|n|}$ [51]. On the other hand the black hole solution carrying these charges is identical to a supersymmetric black hole solution carrying the same charges in the parent type IIA theory before the orientifold projection, and has an entropy $2\pi\sqrt{Q_1Q_5\tilde{Q}_5|n|}$ in the supergravity approximation [51]. Thus the statistical entropy is in agreement with the macroscopic black hole entropy. As we will discuss in § 4.3, in the M-theory description utilized in [52] we can generalize this heuristic counting to a larger class of black holes.

Even though we have considered a four-charge system with a specific charge assignment for simplicity of discussion, the conclusion can be stated in a duality invariant way. Consider first the case of heterotic on $T^4 \times T^2$. The U-duality group in this case is $O(6, 22, Z) \times SL(2, Z)$. Since we are dealing with large black holes and supergravity action without higher derivative corrections, we in fact have $O(6, 22, R) \times SL(2, R)$ at our disposal. Let Q be the electric charges and P be the magnetic charges of the black hole; they are both vectors of $O(6, 22, R)$. Then the black hole entropy in the supergravity approximation can be written in a U-duality invariant way as $S = \pi\sqrt{|P^2Q^2 - (P \cdot Q)^2|}$ where the dot product is the $O(6, 22, R)$ invariant one. For the specific configuration considered earlier we have $Q \cdot P = 0$, $Q^2 = 2Q_1n$ and $P^2 = 2Q_5\tilde{Q}_5$. Note in particular that for our non-supersymmetric state Q^2 is negative since n is negative. A supersymmetric configuration on the other hand would have Q^2 positive. More generally, for a general charge assignment the supersymmetric black holes have the discriminant $P^2Q^2 - (P \cdot Q)^2$ positive and the non-supersymmetric black holes have $P^2Q^2 - (P \cdot Q)^2$ negative in our conventions. The absolute value of the discriminant is the one that enters into the expression for the entropy.

For the heterotic string on $K3 \times T^2$, one obtains an N=2 supergravity in four dimensions. The invariance of the classical supergravity action in this case is $O(2, n_v - 1, R) \times SL(2, R)$ where n_v is the number of N=2 vector multiplets. The formulae above apply with the only change that the dot product is now the $O(2, n_v - 1, R)$ invariant one.

4.3 General extremal black Holes in M-theory on $CY_3 \times S^1$

We will now consider a more general class of examples involving black hole solutions in M-theory compactified on a circle S^1 times a Calabi-Yau 3-fold CY_3 . By the usual duality between type IIA string theory and M-theory on S^1 , these can also be regarded as black hole solutions in type IIA string theory on CY_3 . We will consider the BPS black holes discussed in [52] with vanishing D6-brane charge but arbitrary D4-brane charges $\{p^A\}$, D2-brane charges $\{q_A\}$ and D0-brane charge q_0 . Here the index $A = 1, 2, \dots, n_v$ labels the n_v 4-cycles (or equivalently the dual 2-cycles) of CY_3 . Thus we have p^A D4-branes wrapped on the A -th 4-cycle Σ_A , q_A D2-branes wrapped on the A -th 2-cycle σ^A and q_0 D0-branes. If we denote by P the four cycle $p^A\Sigma_A$, then in the M-theory description this configuration corresponds to a M5-brane wrapped on $P \times S^1$, with appropriate fluxes turned on the brane to produce the D2-brane charges, and carrying q_0 units of momentum along S^1 . If \mathbf{P} is a ‘very ample’ divisor, then it is smooth at a generic point in the moduli space

and an M5-brane wrapped on it is locally a single, smooth brane. Its massless fluctuation modes can then be computed using index theory as in [52] and is summarized by a $(0, 4)$ superconformal field theory living on the effective string wrapping the M-theory circle S^1 . The number of massless left-moving and right-moving bosons and fermions on this string deduced from index theory gives us the left-moving and right-moving central charges c_L and c_R of the conformal field theory (CFT).

Let us denote by N_L^B the left-moving bosons and by N_R^B and N_R^F the right-moving bosons and fermions respectively. Also let $6D_{ABC} = \Sigma_A \cap \Sigma_B \cap \Sigma_C$ be the intersection numbers of the four cycles Σ_A . The central charges are then given by [52]

$$c_L = N_L^B = 6D + c_2 \cdot P \tag{4.1}$$

$$c_R = N_R^B + \frac{1}{2}N_R^F = 6D + \frac{1}{2}c_2 \cdot P, \tag{4.2}$$

where $D = D_{ABC}p^A p^B p^C$ and $c_2 \cdot P \equiv c_{2A}p^A$, c_{2A} being the second Chern class of the four cycle Σ_A . On the other hand the conformal weight of the lowest energy state carrying the charges described above is given by

$$\begin{aligned} (h_L, h_R) &= (\hat{q}_0, 0) \quad \text{for } \hat{q}_0 > 0 \\ &= (0, -\hat{q}_0) \quad \text{for } \hat{q}_0 < 0, \end{aligned} \tag{4.3}$$

where

$$\hat{q}_0 = q_0 + \frac{1}{12}D^{AB}q_Aq_B, \tag{4.4}$$

D^{AB} being the inverse of $D_{AB} \equiv D_{ABC}p^C$. Then according to Cardy formula the statistical entropy, defined as the logarithm of the degeneracy of states, is given by

$$\begin{aligned} S_{\text{stat}} &= 2\pi\sqrt{\frac{c_L h_L}{6}} = 2\pi\sqrt{\left(D + \frac{1}{6}c_2 \cdot P\right)\hat{q}_0} \quad \text{for } \hat{q}_0 > 0 \\ &= 2\pi\sqrt{\frac{c_R h_R}{6}} = 2\pi\sqrt{\left(D + \frac{1}{12}c_2 \cdot P\right)|\hat{q}_0|} \quad \text{for } \hat{q}_0 < 0. \end{aligned} \tag{4.5}$$

The states with $\hat{q}_0 > 0$ are BPS, whereas states with $\hat{q}_0 < 0$ break all supersymmetries.⁵ Since D is cubic in the charges p^A whereas $c_2 \cdot P$ is linear in these charges, we have $D \gg |c_2 \cdot P|$. Thus for both signs of \hat{q}_0 the leading contribution to the statistical entropy is given by

$$S_{\text{stat}} = 2\pi\sqrt{D|\hat{q}_0|}. \tag{4.6}$$

The macroscopic entropy of the corresponding black hole solution to leading order in supergravity goes as [52]

$$S_{\text{BH}} = 2\pi\sqrt{D|\hat{q}_0|}, \tag{4.7}$$

⁵As explained in [52], it is possible to maintain supersymmetry even with right-moving momentum as long as it is a multiple of the integral class $[P]$ in the momentum lattice. But a generic right-moving momentum will break supersymmetry.

for both signs of \hat{q}_0 . This approximation is valid for large charges. Thus the statistical entropy (4.5) agrees with the macroscopic entropy calculated in the supergravity approximation in the large charge limit both for BPS as well as non-BPS states. In fact in this case there are general arguments that this agreement continues to hold for both BPS and non-BPS states even after inclusion of higher derivative corrections [38, 37, 53]. We will return to this point in § 7.

Other examples involving rotating black holes will be discussed in § 8.

5. Geometry of the D1-D5- \bar{P} system

In this section we will analyze in detail the five dimensional black-hole with three charges discussed in § 4.1. For definiteness we will take the compact space to be $T^4 \times S^1$, but the extension to the $K3 \times S^1$ case is straightforward. In the type I description this state couples only to the graviton G_{MN} and the dilaton ϕ from the NS-NS sector, and the 2-form potential B_{MN} from the R-R sector. The low energy action for these fields is

$$S = \int d^{10}x \sqrt{-\det G} \mathcal{L},$$

$$\mathcal{L} = \frac{1}{16\pi G_{10}} \left[e^{-2\phi} (R + 4(\nabla\phi)^2) - \frac{1}{12} H^2 \right], \quad (5.1)$$

where H is the 3-form field strength associated with B_{MN} and

$$16\pi G_{10} = (2\pi)^7 (\alpha')^4 \quad (5.2)$$

would be the ten dimensional Newton's constant if ϕ vanishes asymptotically. The solution with three charges Q_1 , Q_5 , and n with $n < 0$ is the same as the corresponding solution in type-IIB theory [18]

$$dS^2 = \left(1 + \frac{r_1^2}{r^2}\right)^{-1/2} \left(1 + \frac{r_5^2}{r^2}\right)^{-1/2} \left[-dt^2 + dx_5^2 + \frac{r_n^2}{r^2} (dt - dx_5)^2 + \left(1 + \frac{r_1^2}{r^2}\right) dx_i dx^i \right]$$

$$+ \left(1 + \frac{r_1^2}{r^2}\right)^{1/2} \left(1 + \frac{r_5^2}{r^2}\right)^{1/2} [dr^2 + r^2 d\Omega_3^2] \quad (5.3)$$

$$H \equiv \frac{1}{6} H_{MNP} dx^M \wedge dx^N \wedge dx^P = 2\lambda^{-1} r_5^2 \epsilon_3 + 2r_1^2 \lambda e^{-2\phi} *_6 \epsilon_3,$$

$$e^{-2\phi} = \lambda^{-2} \left(1 + \frac{r_5^2}{r^2}\right) \left(1 + \frac{r_1^2}{r^2}\right)^{-1}, \quad (5.4)$$

where x^5 is the coordinate of a circle S^1 with coordinate radius R , x^i for $i = 6, \dots, 9$ are the coordinates of a torus T^4 with coordinate volume $(2\pi)^4 V$, ϵ_3 is the volume element on the unit three-sphere and $*_6$ denotes the Hodge dual in the six dimensions spanned by x^0, \dots, x^5 . Thus λ , $(2\pi)^4 V$, and R are asymptotic values of the string coupling constant, the volume of T^4 , and the radius of the S^1 , all measured in string units. This solution represents a black hole in the five dimensional theory spanned by x^0, \dots, x^4 . The parameters of the solution are related to the integral charges Q_1 , Q_5 and n through the relations

$$r_1^2 = \frac{\lambda Q_1 \alpha'}{V}, \quad r_5^2 = \lambda Q_5 \alpha', \quad r_n^2 = \frac{\lambda^2 |n| \alpha'}{R^2 V}. \quad (5.5)$$

The term involving $(dt - dx_5)^2$ in the metric (5.3) corresponds to right-moving momentum $n < 0$ along the soliton and the solution breaks all supersymmetries. If we instead use $n > 0$ at asymptotic infinity then the solution depends on the combination $(dt + dx_5)^2$ and supersymmetry is preserved.

It is instructive to study the near horizon geometry of this black hole. For this we will set $\alpha' = 1$ and define new coordinates and parameters

$$\begin{aligned}
 \rho &= r^2/(r_n^2 R^2), & \tau &= 2r_n R^2 t/(r_1 r_5), \\
 y^5 &= (x^5 - t)/R, & y^i &= x^i/V^{1/4} \quad \text{for } 6 \leq i \leq 9, \\
 v_1 &= \frac{r_1 r_5}{4} = \frac{1}{4} \frac{\lambda}{\sqrt{V}} \sqrt{Q_1 Q_5}, & v_2 &= r_1 r_5 = \frac{\lambda}{\sqrt{V}} \sqrt{Q_1 Q_5}, \\
 u_1 &= \frac{r_n^2 R^2}{r_1 r_5} = \frac{\lambda}{\sqrt{V}} \frac{|n|}{\sqrt{Q_1 Q_5}}, & u_2 &= \frac{r_1 V^{1/2}}{r_5} = \sqrt{\frac{Q_1}{Q_5}}, \\
 u_3 &= \frac{r_5}{r_1 \lambda} = \frac{\sqrt{V}}{\lambda} \sqrt{\frac{Q_5}{Q_1}}, \\
 e_1 &= \frac{r_5 r_n R}{4 \lambda r_1} = \frac{1}{4} \sqrt{\frac{Q_5 |n|}{Q_1}}, & e_2 &= -\frac{r_1 r_5}{4 r_n R} = -\frac{1}{4} \sqrt{\frac{Q_1 Q_5}{|n|}},
 \end{aligned} \tag{5.6}$$

$$(5.7)$$

and then take the $r \rightarrow 0$ limit. With this definition y^5 has coordinate radius 1, y^6, \dots, y^9 have coordinate volume $(2\pi)^4$ and the near horizon geometry takes the form:

$$\begin{aligned}
 dS^2 &= v_1 \left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + v_2 d\Omega_3^2 + u_1 (dy^5 - 2e_2 \rho d\tau)^2 + u_2 dy^i dy^i \\
 H &= 2 Q_5 \epsilon_3 + 2 e_1 d\tau \wedge d\rho \wedge dy^5, \quad e^{-2\phi} = u_3^2.
 \end{aligned} \tag{5.8}$$

From this we see that many of the fields in this geometry get attracted to fixed values at the horizon. For example, the volume of the T^4 at the horizon gets attracted to $(2\pi)^4 u_2^2 = (2\pi)^4 Q_1/Q_5$ independent of the asymptotic value V . Not all moduli get fixed, however. For example, several parameters including the dilaton at the horizon continue to depend on the asymptotic modulus V/λ^2 . The entropy is, of course, independent of all asymptotic moduli and depends only on charges as $2\pi \sqrt{Q_1 Q_5 \bar{P}}$.

We will now derive the near horizon geometry given in eqs. (5.7), (5.8) using the entropy function formalism [54]. For arbitrary parameters $v_1, v_2, u_1, u_2, u_3, e_1, e_2$ and p_1 , eq.(5.8) describes the general background with zero Kaluza-Klein monopole charge associated with the y^5 direction, preserving the $SO(2,1) \times SO(4)$ symmetry of $AdS_2 \times S^3$. In order to compute the entropy function for this black hole we introduce normalized charges

$$p^1 = Q_5, \quad q_1 = 2 Q_1, \quad q_2 = 2 n, \tag{5.9}$$

and define⁶

$$f(\vec{v}, \vec{u}, \vec{e}, \vec{p}) = \int_H \sqrt{-\det G} \mathcal{L} \tag{5.10}$$

⁶Since the dimensional reduction on $T^4 \times S^1$ produces Chern-Simons terms, eq. (5.10) is not valid in

evaluated in the background (5.8). Here \int_H denotes integration over the horizon of the black hole. In the ten dimensional description this is $S^3 \times S^1 \times T^4$ labeled by the angular coordinates labeling the three sphere, the coordinate y^5 and the coordinates y^6, \dots, y^9 . The entropy function $\mathcal{E}(\vec{v}, \vec{u}, \vec{e}, \vec{q}, \vec{p})$ is then given by [21]

$$\mathcal{E}(\vec{v}, \vec{u}, \vec{e}, \vec{q}, \vec{p}) = 2\pi (q_1 e_1 + q_2 e_2 - f(\vec{v}, \vec{u}, \vec{e}, \vec{p})) , \tag{5.11}$$

and the entropy of the extremal black hole for a given set of electric charges (\vec{q}, \vec{p}) is obtained by extremizing the entropy function with respect to the variables v_i, u_i and e_i .

In the present problem the function f can be easily evaluated and is given by

$$f(\vec{v}, \vec{u}, \vec{e}, \vec{p}) = \frac{4\pi^6}{G_{10}} v_1 v_2^{3/2} \sqrt{u_1} u_2^2 [u_3^2 (-2v_1^{-1} + 6v_2^{-1} + 2u_1 v_1^{-2} e_2^2) + 2u_1^{-1} v_1^{-2} e_1^2 - 2v_2^{-3} p_1^2] . \tag{5.12}$$

This gives, using $G_{10} = 8\pi^6 (\alpha')^4 = 8\pi^6$,

$$\mathcal{E}(\vec{v}, \vec{u}, \vec{e}, \vec{q}, \vec{p}) = 4\pi \left[Q_1 e_1 + n e_2 - \frac{1}{4} v_1 v_2^{3/2} \sqrt{u_1} u_2^2 \left\{ u_3^2 (-2v_1^{-1} + 6v_2^{-1} + 2u_1 v_1^{-2} e_2^2) + 2u_1^{-1} v_1^{-2} e_1^2 - 2v_2^{-3} Q_5^2 \right\} \right] , \tag{5.13}$$

where we have used (5.2) and replaced q_1, q_2 and p_1 in terms of Q_1, n and Q_5 using (5.9). It is easy to see that this function has an extremum at

$$\begin{aligned} v_1 &= \frac{1}{4} \xi \sqrt{Q_1 Q_5}, & v_2 &= \xi \sqrt{Q_1 Q_5}, & u_1 &= \xi \frac{|n|}{\sqrt{Q_1 Q_5}}, & u_2 &= \sqrt{\frac{Q_1}{Q_5}}, \\ u_3 &= \xi^{-1} \sqrt{\frac{Q_5}{Q_1}}, & e_1 &= \frac{1}{4} \sqrt{\frac{Q_5 |n|}{Q_1}}, & e_2 &= -\frac{1}{4} \sqrt{\frac{Q_1 Q_5}{|n|}}, \end{aligned} \tag{5.14}$$

for $n < 0$. Here ξ is an arbitrary parameter reflecting a flat direction of the entropy function. This agrees with (5.11) for $\xi = \lambda/\sqrt{V}$. Furthermore the value of \mathcal{E} evaluated at this extremum is

$$\mathcal{E} = 2\pi \sqrt{Q_1 Q_5 |n|} . \tag{5.15}$$

This reproduces the entropy of this black hole.

The same conclusions can also be reached using the effective potential described in [20, 22]. One finds that the effective potential is extremized for the values of the moduli given in eq. (5.14). The extremum has one flat direction and is a minimum along the two other directions in moduli space. This shows that the extremum is an attractor along the two non-flat directions. For the supersymmetric case the attractor behavior is expected. In

general. One needs to first express the dimensionally reduced Lagrangian density in a manifestly covariant form by throwing away total derivative terms, and then define $f(\vec{v}, \vec{u}, \vec{e}, \vec{p})$ using this covariant Lagrangian density. Typically this gives rise to additional contribution to f besides (5.10) [55]. However in the present example the Chern-Simons terms in H vanish and hence (5.10) gives the correct contribution to f .

the non-supersymmetric case it follows from an invariance of the effective potential under the charge conjugation symmetry, $n \rightarrow -n$.

Now from eq. (5.14) we see that as long as Q_1 , Q_5 and n are large, – say $Q_1 \sim N$, $Q_5 \sim N$, $|n| \sim N^2$ with N large. — all scalars constructed out of curvature and gauge field strengths at the horizon are small for finite ξ . Thus the supergravity approximation is reliable. Furthermore, assuming that the basic symmetry of the attractor geometry does not change from $AdS_2 \times S^3$, one can evaluate the entropy function of [21, 32] to find that the higher derivative terms give subleading corrections. Since the attractor values of the scalars are determined by minimizing the entropy function and the Bekenstein-Hawking-Wald entropy is the value of this function at the minimum, the resulting entropy will have a sensible perturbative expansion in inverse powers of N . Furthermore, since the entropy is independent of ξ , the answer (5.15) will continue to be valid even if ξ is small. In particular, for the scaling of Q_1 , Q_5 and $|n|$ given above if we want the effective interaction strength in the microscopic theory to be small. we need to take ξN to be a small number. In this region v_1 , v_2 and u_1 are small indicating that the higher derivative corrections become important. Nevertheless our argument shows that the Wald entropy will continue to be given by (5.15).

The argument given above assumes that the flat direction labeled by ξ is not lifted when we add higher derivative terms to the action. For supersymmetric black holes, — e.g. the one obtained by replacing $n \rightarrow -n$, $e_2 \rightarrow -e_2$ in the solution described above, — we expect this to be true. As a result the value of ξ at the horizon is a free parameter and the value of the entropy is independent of this parameter. However for the non-supersymmetric black holes the flat directions may get lifted under addition of higher derivative corrections at some order.⁷ In that case the parameter ξ appearing in (5.14) will no longer be arbitrary and will take some fixed value independent of the asymptotic moduli. As long as ξN at the fixed point is large the horizon geometry has low curvature and higher derivative corrections are small. However if ξN becomes of order one or less, we have highly curved horizon geometry and the derivative expansion is no longer sensible for the computation of the entropy function.⁸ In § 6 we will give a uniform treatment of all these cases by introducing a small amount of non-extremality to control the effect of non-trivial dependence of the entropy function on ξ .

So far most of our attention has been focussed on the near horizon geometry. Let us now look closely at the full interpolating geometry given in (5.3), (5.4). First consider the case $\lambda N \gg 1$. In this case r_1 , r_5 and r_n are large. The near horizon geometry $AdS_2 \times S^3$ is obtained if we can “drop the one” in the harmonic functions appearing in the

⁷In the heterotic description the parameter ξ^{-2} actually correspond to the volume of the T^4 measured in the string metric. Since there are no charges associated with the gauge field arising out of T^4 compactification, the full black hole geometry is a product space of T^4 and a six dimensional manifold labeled by x^0, \dots, x^5 . This explains why in the supergravity approximation the modulus ξ does not get fixed by the attractor mechanism. In fact this feature continues to hold even after inclusion of tree level higher derivative corrections in the heterotic string theory. The loop corrections however will couple the black hole geometry and the six dimensional geometry and is expected to generate a potential for ξ .

⁸Note that since now ξ at the horizon is independent of the asymptotic coupling, this problem exists even when the asymptotic 't Hooft coupling is large.

equations (5.3) and (5.4). This can be done once $r \ll r_1, r_5, r_n$ simultaneously. Since r_1, r_5 and r_n are large we can “drop the one” even if r remains large compared to the string scale and one never runs into a high curvature region in the interpolating geometry all the way from the asymptotic infinity to the horizon. In this regime, higher derivative corrections to the solution are small throughout the entire geometry. Now consider what happens when we start reducing the asymptotic coupling λ keeping N fixed at some large value. Once λN becomes of order 1, the radii r_1, r_5 are no longer large and in order to reach the near horizon geometry we need to take $r \ll 1$. Thus the geometry enters the large curvature region $r \sim 1$. In this region corrections to the action due to higher derivative terms are no longer small and we do not have a systematic approximation scheme for calculating these corrections. Nevertheless, as long as the solution approaches the $AdS_2 \times S^3$ form given in (5.8) for small r , the near horizon geometry and entropy are determined by extremizing the entropy function and as a result the entropy is equal to its value for $\lambda N \gg 1$ as long as we can ignore the issue of lifting of the flat direction.

6. Taming the runaway

In this section we will address the potential problem with the runaway behavior of near horizon parameters after inclusion of higher derivative corrections to the supergravity action. In particular, we are interested in a situation where the leading two-derivative action gives rise to a flat direction of the entropy function or equivalently the effective potential. In such a case, the higher derivative corrections to the entropy function could lift the flat directions in such a way that the entropy function has no extremum. This would result in runaway behavior. What is the meaning of the entropy calculated in the leading two-derivative approximation in such a situation? In answering this question it is useful to regard the entropy of the extremal black hole as a limit of the entropy of a non-extremal black hole. By taking a slightly non-extremal black hole, and large enough charge, we will see below that the run-away behavior is in effect “cut-off”. Since the black hole is only slightly non-extremal the entropy would be close to that of the extremal case calculated in the two-derivative approximation.

Even though we will discuss the issue in the context of four dimensional examples, the analysis easily generalizes to other dimensions. We use the notation of [22] and consider a theory with a lagrangian density of the form

$$\sqrt{-\det g} \mathcal{L} = \frac{1}{\kappa^2} \left[R - 2 g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - f_{ab}(\vec{\phi}) F_{\mu\nu}^a F^{b\mu\nu} - \frac{1}{2} (\sqrt{-\det g})^{-1} \tilde{f}_{ab}(\vec{\phi}) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^b \right], \quad (6.1)$$

where $g_{\mu\nu}$ denotes the metric, $\{\phi_i\}$ denote a set of neutral scalar fields and $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ denote a set of gauge field strengths. $f_{ab}(\phi)$ and $\tilde{f}_{ab}(\vec{\phi})$ are a set of functions which are fixed for a given theory. In this theory we look for a spherically symmetric black hole

solution of the form:⁹

$$\begin{aligned} ds^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu = -a(r)^2 dt^2 + a(r)^{-2} dr^2 + b(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2), \\ \frac{1}{2} F_{\mu\nu}^a dx^\mu \wedge dx^\nu &= Q_{(m)}^a \sin \theta d\theta \wedge d\phi + f^{ab}(\vec{\phi}) (Q_{(e)b} - \tilde{f}_{bc}(\vec{\phi}) Q_{(m)}^c) b(r)^{-2} dt \wedge dr \\ \phi_i &= \phi_i(r), \end{aligned} \tag{6.2}$$

where $f^{ab}(\vec{\phi})$ is the matrix inverse of $f_{ab}(\vec{\phi})$, $Q_{(m)}^c$ and $Q_{(e)c}$ denote respectively the magnetic and electric charges associated with the gauge field A_μ^c , and $a(r)$, $b(r)$ and $\phi_i(r)$ are functions to be determined. The equations determining the radial evolution of $a(r)$, $b(r)$ and $\phi_i(r)$ can be derived from a one dimensional lagrangian [22]

$$\frac{2}{\kappa^2} \int dr \left[(a^2 b)' b' - a^2 b^2 (\phi')^2 - b^{-2} V_{\text{eff}}(\vec{\phi}) \right] \tag{6.3}$$

where prime denotes derivative with respect to the radial variable r and

$$V_{\text{eff}}(\vec{\phi}) = f^{ab} (Q_{(e)a} - \tilde{f}_{ac} Q_{(m)}^c) (Q_{(e)b} - \tilde{f}_{bd} Q_{(m)}^d) + f_{ab} Q_{(m)}^a Q_{(m)}^b. \tag{6.4}$$

If $V_{\text{eff}}(\vec{\phi})$ has a minimum at $\vec{\phi} = \vec{\phi}_0$ with $V_{\text{eff}}(\vec{\phi}_0) = Q^2$, and if we parametrize the radial coordinate r in such a way that the horizon is at $r = Q$, then for an extremal black holes, as $r \rightarrow Q$ we have [22]

$$\vec{\phi}(\vec{r}) \rightarrow \vec{\phi}_0, \quad a(r) \rightarrow \frac{r-Q}{Q}, \quad b(r) \rightarrow Q. \tag{6.5}$$

This describes an $AdS_2 \times S^2$ near horizon geometry.

The effective potential $V_{\text{eff}}(\vec{\phi})$ typically has some flat directions and hence a family of minima. At the minima some moduli χ_α are fixed to be χ_α^* , but some moduli, representing deformations along these flat directions, are not fixed. For simplicity we consider the case where there is only one such flat direction and label the coordinate along this direction by ξ . An example of such a flat direction is provided by the case discussed in § 5, where the flat direction is also called ξ .

For simplicity, in the analysis below we will set the asymptotic values of all the moduli χ_α to their attractor values χ_α^* , so that in the leading supergravity approximation these moduli remain constant for all r : $\chi_\alpha(r) = \chi_\alpha^*$. In this approximation the ξ modulus is also independent of r since the effective potential is ξ -independent. With these boundary conditions, the leading effective potential evaluated on the solution is a constant, independent of r . It is also independent of the flat direction ξ . So we write

$$V_{\text{eff}}|_{\text{solution}} = Q^2, \tag{6.6}$$

where Q is a constant independent of r . The resulting solution is the extremal Reissner-Nordstrom black hole,

$$a^2 = (1 - Q/r)^2 \quad b = r. \tag{6.7}$$

⁹In (6.2) we have fixed the form of the gauge field strengths by requiring that they solve the Bianchi identities and field equations.

Note that in our conventions the parameter Q has dimension of length. There are also non-extremal black holes. These have,

$$a^2 = \left(1 - \frac{\alpha}{r}\right) \left(1 - \frac{\beta}{r}\right), \quad b = r, \quad (6.8)$$

with

$$\alpha\beta = Q^2. \quad (6.9)$$

We take $\alpha > \beta$ by convention, so that the outer horizon is at $r = \alpha$. A slightly non-extremal black hole has,

$$\frac{(\alpha - \beta)}{\alpha} \ll 1. \quad (6.10)$$

Let us now ask what happens when higher derivative terms contribute an extra term $h(\xi)$ in V_{eff} so that $V_{\text{eff}}(\vec{\phi})$ has the form

$$V_{\text{eff}}(\vec{\phi}) = f^{ab}(Q_{(e)a} - \tilde{f}_{ac}Q_{(m)}^c)(Q_{(e)b} - \tilde{f}_{bd}Q_{(m)}^d) + f_{ab}Q_{(m)}^a Q_{(m)}^b + h(\xi). \quad (6.11)$$

The resulting one dimensional action is then,

$$S = \frac{2}{\kappa^2} \int dr \left((a^2 b)' b' - a^2 b^2 (\xi')^2 - \frac{V_{\text{eff}}}{b^2} \right), \quad (6.12)$$

with V_{eff} given in eq. (6.11). $h(\xi)$, having its origin in four and higher derivative terms in the effective action, is of order Q^{-k} with $k \geq 0$ for $\xi \sim 1$.¹⁰

We will consider a non-extremal black hole and will self consistently solve the equations by assuming that ξ does not vary significantly from its asymptotic value at $r = \infty$ all the way till the horizon of the black hole.¹¹ Consistent with this assumption, to leading order $\xi(r) = \xi(\infty)$. As long as $\xi(\infty)$ is of order one, the effective potential is approximately

$$V_{\text{eff}} \simeq Q^2 + h(\xi(\infty)).$$

To this order the metric of a slightly non-extremal black hole is then given by eq. (6.8) with

$$\alpha\beta = Q^2 + h(\xi(\infty)). \quad (6.13)$$

¹⁰We could include a dependence of h on the other moduli fields $\{\chi_\alpha\}$, but this will not affect our main conclusions. Also, strictly speaking if the additional terms are arising due to higher derivative corrections, we need to keep other higher derivative terms in the analysis, for example, in the kinetic energy terms for scalars etc. In general after inclusion of these terms the equations of motion will have more solutions some of which could diverge at the horizon. We are assuming that if we choose the solution that is regular at the horizon then it can be matched on to the asymptotically flat Minkowski space-time. We expect that for such solutions the effect of these higher derivative terms will remain small all through the solution and will not change our main conclusion that for big enough Q^2 , if the black hole is only slightly non-extremal, ξ essentially does not evolve from its value at ∞ all the way to the horizon.

¹¹It is not necessary to consider the evolution all the way from ∞ to the horizon. In particular when the asymptotic coupling constant is small, we expect that the curvature and other field strengths will become large in an intermediate region where the higher derivative terms play an important role. Nevertheless the geometry is expected to emerge into an approximately $AdS_2 \times S^2$ geometry sufficiently close to the horizon. We can then concentrate on the radial evolution of ξ in this region, and show that ξ does not vary appreciably in this region.

We now turn to calculating the radial evolution of ξ . Since the only ξ dependence of $V_{\text{eff}}(\vec{\phi})$ comes from the $h(\xi)$ term in (6.11), ξ satisfies the equation,

$$\partial_r(a^2 b^2 \partial_r \xi) = \frac{g(\xi)}{2b^2} \tag{6.14}$$

where

$$g(\xi) = \partial_\xi h(\xi). \tag{6.15}$$

In writing down (6.14) we have assumed that ξ is a canonically normalized field. To calculate the first corrections we will set $\xi = \xi(\infty)$ on the right hand side of (6.14) and then solve this equation. This gives,

$$\xi(r) = \frac{g(\xi(\infty))}{2\alpha\beta} \ln\left(\frac{r-\beta}{r}\right) + \xi(\infty). \tag{6.16}$$

In arriving at (6.16) we have fixed an integration constant so that the solution is non-singular at the horizon $r = \alpha$. Indeed, from (6.16) we see that $\xi(r)$ approaches a finite limit as we approach the horizon $r = \alpha$. If however we take the extremal limit when $\alpha = \beta$, $\xi(r)$ has a runaway behavior as we approach the horizon unless $g(\xi(\infty)) = 0$. In the full solution this condition takes the form $g(\xi(\alpha)) = 0$, i.e. ξ should approach an extremum of $h(\xi)$ as we approach the horizon. If $h(\xi)$ does not have an extremum then there is no way to avoid the runaway behavior.

Let us now return to the case of a near extremal black hole. For our approximation to be self consistent, we need $\xi(\alpha) \simeq \xi(\infty)$. More generally we require $\xi(r)$ in the whole range between α and ∞ to be close to $\xi(\infty)$. This means, from eq. (6.16),

$$\left| \ln\left(\frac{\alpha-\beta}{\alpha}\right) \right| \ll \left| \frac{2\alpha\beta}{g(\xi(\infty))} \right|. \tag{6.17}$$

Using the leading order result (6.9) $\alpha\beta$ on the right hand side of eq. (6.17) can be approximated by Q^2 . Using eqs. (6.10), (6.17) we now get

$$1 \gg \frac{\alpha-\beta}{\alpha} \gg e^{-\left| \frac{2Q^2}{g(\xi(\infty))} \right|} \tag{6.18}$$

As long as $g(\xi(\infty)) \sim 1$, the term on the right hand side of (6.18) is exponentially suppressed for large Q . Thus the condition eq. (6.18) can be easily met by appropriate choice of the non-extremality parameter. When this condition is met, the entropy of the non-extremal black hole is approximately given by,

$$S_{\text{BH}} \simeq \pi Q^2 \tag{6.19}$$

which is the entropy of the extremal black hole in the leading approximation. However since V_{eff} receives correction proportional to $h(\xi(\infty))$, we expect that the entropy also receives a similar correction. Since this clearly depends on the asymptotic value $\xi(\infty)$ of the field ξ , we see that for non-extremal black holes of this type, the attractor behavior breaks down at the order in which the potential for ξ is generated.

We note in passing that in any case the entropy of an extremal black hole should be defined by extrapolating the answer from the non-extremal case down to the extremal case, since sufficiently close to extremality the thermal description breaks down and a direct analysis based on thermodynamics becomes unreliable [56]. For the thermal description to work, we need that $(\partial T/\partial M) \ll 1$ where M is the mass of the black hole. This gives rise to the condition $(\alpha - \beta)/\alpha \gg l_{pl}^2/Q^2$. This is a stronger restriction than (6.18) when Q is large. Thus as the non-extremality parameter $(\alpha - \beta)/\alpha$ is reduced, the thermal description will break down before any appreciable running of ξ field can occur outside the horizon. Since the usual Bekenstein-Hawking entropy (and presumably its Wald generalization) of the extremal black hole is obtained by extrapolating the answer obtained at the stage when the thermal description is still reliable, we see that the running of the modulus ξ plays no appreciable role if the entropy is obtained using this procedure.

In summary, for a black hole which is close but not very close to extremality, one finds that the modulus ξ does not evolve an appreciable amount outside the horizon. The entropy of the resulting black hole is close to that of the extremal one obtained by keeping the leading term in the effective potential as long as ξ is of order 1. This however is not the end of the story. In order to argue that the black hole entropy remains approximately constant up to the region of parameter space where the microscopic description is good, we may need to continue the parameter ξ into a region where the near horizon geometry develops large curvature and hence the function $h(\xi)$ becomes comparable to or larger than the leading term. Can we argue that this does not happen? As discussed earlier, one needs to address this question on a case by case basis. We will illustrate this in the context of the example described in § 5. For $Q_1 \sim N$, $Q_5 \sim N$ and $|n| \sim N^2$ with N large, the microscopic description is good when $\lambda N \ll 1$. For $V \sim 1$ this requires $\xi N \ll 1$. Examining the near horizon geometry given in (5.14) we see that in this region the sizes of AdS_2 and S^2 become small, and hence α' corrections in the type I description become important. On the other hand the type I string coupling constant is exceedingly small and hence we can ignore the loop corrections. Thus the question is: do the α' corrections generate a contribution to $h(\xi)$? To answer this question note that the α' corrections in type I theory are the same as those in the parent type IIB theory before the orientifold projection. Since the corresponding black hole in the parent type IIB theory is supersymmetric, we expect that in this case the near horizon value of ξ is arbitrary. Thus the same will hold true for the α' corrected type I theory. This in turn shows that $h(\xi)$ does not receive any contribution due to α' correction.

Finally we note that even in situations where ξ is not a runaway direction and the full entropy function does have an extremum as a function of ξ , we can still regulate the evolution of ξ in the AdS_2 throat using the trick described in this section.¹² By introducing a small non-extremality parameter we can ensure that ξ at the horizon does not change by an appreciable amount from its asymptotic value. The entropy of such black holes remain close to the one found in the leading approximation, and hence can be computed reliably using the entropy function method.

¹²In fact, we may be forced to do this to make the computation of statistical entropy well defined.

7. Comparison between AdS₂ and AdS₃ based approaches

For some extremal black holes in string theory the AdS_2 component of the near horizon geometry, together with an internal circle, describes a locally AdS_3 space. More specifically the near horizon geometry of these extremal black holes correspond to that of extremal BTZ black holes [57] in AdS_3 with the momentum along the internal circle representing the angular momentum of the black hole [58]. In such situations, alternative arguments are available for explaining the agreement between the leading order thermodynamic and statistical entropy. These arguments are quite powerful and applicable even for non-BPS extremal black holes. In particular the enhanced isometry group of the AdS_3 space allows us to get a more detailed information about the entropy of the system and prove certain non-renormalization theorems [37–39] for the entropy of supersymmetric as well as non-supersymmetric black holes. In this section we will outline these arguments both from macroscopic and microscopic points of view so as to clearly distinguish them from the more general argument presented in this paper, and also carry out a comparison between the two approaches when both methods are available.

The rest of this section is organized as follows. In § 7.1 we review the computation of the macroscopic entropy based on the AdS_3 near horizon geometry and compare the relative strength and weaknesses of the AdS_3 and AdS_2 based approaches. In § 7.2 we give examples of extremal BPS and non-BPS black holes in string theory which do not have any AdS_3 factor so that the arguments of [37–39] cannot be applied directly on such black holes. In § 7.3 we will discuss the microscopic description of black holes with locally AdS_3 near horizon geometry and its implication for the non-renormalization of the statistical entropy of the system.

7.1 Black holes with AdS_3 near horizon geometry

We begin by reviewing the origin of the AdS_3 geometry. For this we focus on the AdS_2 part of the near horizon geometry together with the electric flux through it. By choosing the basis of gauge fields appropriately we can arrange that only one gauge field has non-vanishing electric flux through the AdS_2 ; let us denote this gauge field strength by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then the relevant part of the near horizon background takes the form:

$$ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta = v_1(-r^2 dt^2 + r^{-2} dr^2), \quad F_{rt} = e. \quad (7.1)$$

Let us now assume that there is an appropriate duality frame in which we can regard the gauge field component A_μ as coming from the component of a three dimensional metric along certain internal circle. Let ϕ be the scalar field representing the metric component along the extra circle. Then the three dimensional metric can be expressed in terms of the two dimensional fields as

$$ds_3^2 = \phi \left(g_{\alpha\beta} dx^\alpha dx^\beta + (dy + A_\alpha dx^\alpha)^2 \right) \quad (7.2)$$

where y denotes the coordinate along the circle. For definiteness we will assume that y has period 2π . The relation between the three dimensional metric and the two dimensional

metric given above is somewhat non-standard; this is related to the standard form by a rescaling of the two dimensional metric by ϕ . Since (7.1) gives $A_t = er$, and since the scalar field ϕ must take some constant value u at the horizon, we see that the three dimensional near horizon metric has the form

$$ds_3^2 = u [v_1(-r^2 dt^2 + r^{-2} dr^2) + (dy + er dt)^2]. \quad (7.3)$$

One can show that if v_1 and e satisfy the relation

$$v_1 = e^2, \quad (7.4)$$

then the three dimensional metric (7.3) describes a locally AdS_3 space. Had the coordinate y taken values along a real line, it would be a globally AdS_3 space; however because of the periodic identification we have a quotient of the AdS_3 space by a translation by 2π along y . The effect of taking this quotient is to break the $SO(2,2)$ isometry group of AdS_3 to $SO(2,1) \times U(1)$, — the symmetries of an $AdS_2 \times S^1$ manifold. Since the physical radius of the y circle is given by $\sqrt{G_{yy}} = \sqrt{u}$, we expect that the effect of this symmetry breaking will be small for large u .

Let us for the time being ignore the effect of this symmetry breaking and suppose that the background has full symmetries of the AdS_3 space. In this case we expect that the dynamics of the theory in this background will be governed by an effective three dimensional action, obtained by treating all the other directions, including the azimuthal and polar coordinates ϕ and θ labeling the non-compact part of space, as compact. This effective action will have the form

$$\int d^3x \sqrt{-\det G} (\mathcal{L}_0^{(3)} + \mathcal{L}_1^{(3)}), \quad (7.5)$$

where $\mathcal{L}_0^{(3)}$ is a lagrangian density with manifest general coordinate invariance, and $\sqrt{-\det G} \mathcal{L}_1^{(3)}$ denotes the gravitational Chern-Simons term:

$$\sqrt{-\det G} \mathcal{L}_1^{(3)} = K \Omega_3, \quad (7.6)$$

Ω_3 being the Lorentz Chern-Simons 3-form and K is a constant. One can then show, both in the Euclidean action formalism [38, 37, 59] as well as using Wald's formula [60, 53], that the entropy of the black hole with near horizon geometry described in (7.3) has the form:

$$\begin{aligned} S_{\text{BH}} &= 2\pi \sqrt{\frac{c_L n}{6}} \quad \text{for } n > 0, \\ &= 2\pi \sqrt{\frac{c_R |n|}{6}} \quad \text{for } n < 0, \end{aligned} \quad (7.7)$$

where n is the electric charge associated with the gauge field A_μ , and

$$c_L = 24(-g(l) + \pi K), \quad c_R = 24\pi(-g(l) - \pi K), \quad (7.8)$$

$$g(l) = \frac{1}{4} \pi l^3 \mathcal{L}_0^{(3)}, \quad l = 2\sqrt{ue^2}. \quad (7.9)$$

$\mathcal{L}_0^{(3)}$ in (7.9) has to be evaluated on the near horizon background (7.3). This gives a concrete form of the n dependence of the entropy in terms of the constants c_L and c_R .

The constants c_L and c_R given in (7.8) can be interpreted as the left- and right-moving central charges of the two dimensional CFT living on the boundary of the AdS_3 [38, 37, 59]. $|n|$ has the interpretation of L_0 (or \bar{L}_0) eigenvalue of the state in this CFT, and (7.7) can be interpreted as the Cardy formula in this CFT. This observation by itself does not give any further information about the values of c_L and c_R , but a further simplification occurs if the theory has sufficient number of supersymmetries. If the boundary theory happens to have $(0, 4)$ supersymmetry, then the central charge c_R is related to the central charge of an $SU(2)_R$ current algebra which is also a part of the $(0, 4)$ supersymmetry algebra. Associated with the $SU(2)_R$ currents there will be $SU(2)$ gauge fields in the bulk, and the central charge of the $SU(2)_R$ current algebra will be determined in terms of the coefficient of the gauge Chern-Simons term in the bulk theory. This determines c_R in terms of the coefficient of the gauge Chern-Simons term in the bulk theory [38, 37]. On the other hand from (7.8) we see that $c_L - c_R$ is determined in terms of the coefficient K of the gravitational Chern-Simons term. Since both c_L and c_R are determined in terms of the coefficients of the Chern-Simons term in the bulk theory, they do not receive any higher derivative corrections. This completely determines the entropy from (7.7). Furthermore the expression for the entropy derived this way is independent of all the near horizon parameters and hence also of the asymptotic values of all the scalar fields. Thus the entropy remains unchanged as we go from the ‘strong’ coupling regime to the weak coupling regime.

Clearly the existence of an AdS_3 factor in the near horizon geometry gives us results which are much stronger than the ones which can be derived based on the existence of only an AdS_2 factor. However, as indicated above, these results are valid only if the physical radius of the compact y coordinate is large. Typically near horizon value of the radius of the y direction is fixed by the entropy function extremization conditions (the attractor equations) and is not a free parameter. If the charges carried by the black hole are large but all of the same order then the sizes of the compact directions are also of order unity. In this case we expect the $SO(2,2)$ symmetry of AdS_3 to be broken strongly. As a result the effective two dimensional action governing the dynamics in AdS_2 space, besides having a ‘local’ piece of the form (7.5), contains additional terms which cannot be written as dimensional reduction of a generally covariant three dimensional action. There are various sources of these additional terms, e.g. due to the quantization of the momenta along the y direction, contribution to the effective action from various euclidean branes wrapping the y circle, etc. In the presence of such terms there will be additional contribution to the entropy which are not of the form (7.7). These additional corrections can be interpreted as due to the corrections to the full string theory partition function on thermal AdS_3 [61, 39] or equivalently as corrections to the Cardy formula in the CFT living on the boundary of AdS_3 , but there is no simple way to calculate these corrections without knowing the details of this CFT.

We will illustrate this by an example. We consider heterotic string theory compactified on $T^4 \times S^1 \times \tilde{S}^1$ and consider an extremal dyonic black hole in this theory with n

units of momentum and w units of fundamental string winding along S^1 and \tilde{N} units of Kaluza-Klein monopole charge and \tilde{W} units of H-monopole charge along \tilde{S}^1 . In the leading supergravity approximation the near horizon values of the radii R and \tilde{R} of S^1 and \tilde{S}^1 and field S representing square of the inverse string coupling are given by (see e.g. [32])

$$R = \sqrt{\left| \frac{n}{w} \right|}, \quad \tilde{R} = \sqrt{\left| \frac{\tilde{W}}{\tilde{N}} \right|}, \quad S = \sqrt{\left| \frac{nw}{\tilde{N}\tilde{W}} \right|}. \quad (7.10)$$

Furthermore the entropy is given by

$$S_{\text{BH}} = 2\pi \sqrt{|nw\tilde{N}\tilde{W}|}. \quad (7.11)$$

This clearly has the form given in (7.7) with $c_L = c_R = 6|w\tilde{N}\tilde{W}|$. This is a consequence of the fact that the circle S^1 and the near horizon AdS_2 geometry combines into an AdS_3 space if we treat the coordinate along S^1 as non-compact. Otherwise we get a quotient of the AdS_3 space.

Now from (7.10) we see that if we take $|n|$ large keeping the other charges fixed, the radius R of the circle S^1 becomes large. Thus we expect that in this limit the entropy will have the form given in (7.7) even after inclusion of higher derivative corrections. However when all charges are of the same order then the higher derivative corrections to the action will contain terms which cannot be regarded as the dimensional reduction of a three dimensional general coordinate invariant action of the form given in (7.5), and the higher derivative corrections to the entropy will cease to be of the form given in (7.7). This can be seen explicitly by taking into account the effect of the four derivative Gauss-Bonnet term in the four dimensional effective action describing heterotic string compactification on $T^4 \times S^1 \times \tilde{S}^1$. The lagrangian density has a term of the form:

$$\Delta\mathcal{L} = \sqrt{-\det g} \phi(a, S) (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2), \quad (7.12)$$

where

$$\phi(a, S) = -\frac{3}{16\pi^2} \ln (2S|\eta(a + iS)|^4). \quad (7.13)$$

Here $\eta(\tau)$ is the Dedekind eta function and a denotes the axion field whose near horizon value vanishes for the black hole we are considering. The effect of (7.13) on the black hole entropy can be computed using the entropy function method, and to first order its effect is to give an additive contribution to the entropy of the form $-2\pi\Delta\mathcal{L}$ evaluated in the background (7.10). This gives

$$\Delta S_{\text{BH}} = 64\pi^2 \phi(0, S)|_{S=\sqrt{|nw/\tilde{N}\tilde{W}|}} = -12 \ln \left[2\sqrt{\left| \frac{nw}{\tilde{N}\tilde{W}} \right|} \eta \left(i\sqrt{\left| \frac{nw}{\tilde{N}\tilde{W}} \right|} \right)^4 \right]. \quad (7.14)$$

In the limit of large $|n|$ at fixed values of the other charges, S is large and $\eta(iS) \sim e^{-\pi S/12}$. Thus the leading correction to ΔS_{BH} given in (7.14) goes as

$$4\pi \sqrt{\left| \frac{nw}{\tilde{N}\tilde{W}} \right|}. \quad (7.15)$$

Since this is proportional to $\sqrt{|n|}$ we see that the expression for the entropy retains the form given in (7.7) with some correction terms in c_L, c_R .¹³ However when all the charges are of the same order then S is of order unity and we cannot express the corrected entropy $\mathcal{E} + \Delta\mathcal{E}$ in the form given in (7.7).

It is instructive to study the origin of the terms which break the $SO(2,2)$ symmetry of AdS_3 . First of all (7.14) contains a correction term proportional to $\ln S \sim \ln \left| \frac{nw}{NW} \right|$. This can be traced to the effect of replacing the continuous integral over the momentum along S^1 by a discrete sum. There are also additional corrections involving powers of $e^{-2\pi S}$. These can be traced to the effect of Euclidean 5-branes wrapped on $K3 \times S^1 \times \tilde{S}^1$ [62]. Since the 5-brane has one of its legs along S^1 , it breaks the $SO(3,1)$ isometry of Euclidean AdS_3 .

The above example also illustrates the basic difference between the approximation scheme used by the AdS_3 and AdS_2 based approaches. The AdS_3 based approach is useful when we take the momentum along the AdS_3 circle S^1 to be large keeping the other charges fixed. In this limit the size of S^1 becomes large (see eq. (7.10)) and hence the $SO(2,2)$ symmetry of AdS_3 is broken weakly. As a result the entropy has the form (7.7). In the CFT living on the boundary of AdS_3 , this corresponds to a state with large L_0 (or \bar{L}_0) eigenvalue, keeping the central charge fixed. This is precisely the limit in which the Cardy formula for the degeneracy of states is valid. On the other hand the AdS_2 based approach is useful if all the charges are large since in this limit the AdS_2 has small curvature, and we can use the derivative expansion of the effective action to find a systematic expansion of the entropy and the entropy function in inverse powers of charges.

It is natural to wonder about possible additional contribution to the entropy function from other Euclidean brane configurations, *e.g.* heterotic world-sheet instantons. Since in the supergravity approximation the moduli associated with the T^4 part are not fixed by the attractor mechanism, they can be chosen to have any value that we like. If we take one of the circles of the torus to be of sufficiently small size, then the fundamental heterotic string world-sheet, wrapped on the two dimensional torus spanned by this circle and the circle S^1 that becomes part of AdS_3 , can be made to have arbitrarily small action and could in principle give a large contribution to the effective action. This in turn would break the $SO(3,1)$ symmetry of Euclidean AdS_3 strongly. In this case however it is known that these instantons do not lift the flat directions associated with the moduli of T^4 . Since the near horizon field configuration is obtained by extremizing the entropy function it follows that the entropy function cannot receive any contribution from these world-sheet instantons. As a result the entropy also does not receive any contribution from such corrections. The key point in this argument is that the function whose extremization gives the near horizon geometry also gives the entropy.

We expect this to be a generic situation, namely that even in cases where the near horizon geometry has an AdS_3 factor, for some choices of the undetermined moduli there are potential sources for strong breaking of the $SO(3,1)$ symmetry. One then requires use of non-renormalization theorems which prevent lifting of flat directions associated with

¹³For $n < 0$, i.e. non-supersymmetric extremal black holes, the entropy gets some additional corrections from other higher derivative terms which further corrects the expression for c_R .

these undetermined moduli, together with the fact that the extremization of the entropy function determines all these moduli, to argue that the entropy function does not receive any correction from these $SO(3,1)$ breaking terms.

In the supersymmetric case the correction to the entropy given in (7.14) can be shown to agree with the corresponding result for statistical entropy [63, 64]. It will be important to examine if similar agreement also holds for the non-supersymmetric extremal black holes.

7.2 Black holes without AdS_3 factor

In the examples discussed above the effect of deviation from the AdS_3 geometry shows up in the non-leading order. However the non-leading corrections to the entropy, being the analog of finite size effects, are dependent on the ensemble used to compute the entropy and could introduce an ambiguity in the definition of the entropy. A related phenomenon is the breakdown of the thermal description close to the extremal limit discussed in the paragraph below (6.19). The lower limit on the non-extremality parameter introduced there would give rise to an additional contribution to the entropy depending on the precise value of the non-extremality parameter. Such corrections could mask the higher derivative corrections.¹⁴ Furthermore, for non-supersymmetric black holes with runaway scalars, there is an independent need to consider slightly non-extremal black holes (more discussion on this can be found in § 3 and § 6). For these reasons it will be useful to find examples where the leading solution itself does not have an AdS_3 factor in its near horizon geometry. This will be the subject of study in this section. In these examples the ‘leading solution’ does not necessarily mean the solution in the supergravity approximation. For example some of these examples will involve small black holes whose leading entropy comes from higher derivative terms.

The first example involves M-theory compactified to five dimensions on a Calabi-Yau three-fold. Extremal non-rotating black holes in the five dimensional theory that could be BPS or non-BPS would have near horizon geometry $AdS_2 \times S^3$ and correspond to some microscopic configuration of M2-branes wrapping the 2-cycles of the Calabi-Yau space. In the special situation when the Calabi-Yau space is elliptically fibered with a base B , the theory has a dual description as type IIB compactification on $B \times S^1$, and the S^1 factor can combine with the AdS_2 to produce a locally AdS_3 space [44]. However, in general, one can choose a Calabi-Yau manifold that is not elliptically fibered. In this case there is no duality frame in which the compact space has a circle factor, and the near horizon geometry of extremal black holes in the resulting theory does not have an obvious AdS_3 factor.¹⁵ However one can still use entropy function method to calculate the entropy of

¹⁴In principle these ambiguities are present for both BPS and non-BPS black holes. Nevertheless for the BPS states the comparison between the statistical entropy and black hole entropy has been carried out for corrections which are suppressed by powers of Q^{-2} using a microcanonical ensemble [63, 64]. If we really needed to introduce a non-extremality parameter of order l_{pl}^2/Q^2 in order to be able to calculate the entropy then this would have introduced additional corrections to the entropy which are suppressed by power of Q^{-2} and which depend on the precise value of the non-extremality parameter. In this case precision comparison between the two entropies would not have been possible.

¹⁵It is in principle possible that in some appropriate limit a contractible circle inside the Calabi-Yau space becomes large and combines with the AdS_2 factor to form an approximately AdS_3 space.

these extremal black holes. It is not known at present how to compute the microscopic entropy but our conjecture implies a new prediction that it should equal the macroscopic black hole entropy for both BPS and non-BPS extremal black holes.

Another example of an extremal black hole without AdS_3 factor is an extremal, non-BPS, electrically charged black hole in heterotic string theory in ten dimensions. An elementary string with only right-moving oscillator excitations of level N_R and left-moving charge vector \vec{Q} , satisfying the level matching condition $\vec{Q}^2 = 2N_R + 1$ (in the Neveu-Schwarz sector), describes a state that breaks all supersymmetries. The statistical entropy computed from counting of the degeneracy of states is given by

$$S_{\text{stat}} \simeq 2\sqrt{2}\pi\sqrt{N_R} \simeq 2\pi\sqrt{Q^2} \tag{7.16}$$

for large Q^2 . We expect the supergravity description of this state to be an extremal small black hole. Since there is no physical circle associated with the charge \vec{Q} , there is no underlying AdS_3 geometry. Nevertheless our argument will imply that the microscopic entropy of the system should match the macroscopic entropy associated with the small black hole. In fact from the general scaling argument of [41–43] it follows that the entropy of such a black hole is proportional to $\sqrt{Q^2}$ in agreement with (7.16). It will be interesting to explore if the constant of proportionality agrees with the prediction from the microscopic entropy.

We could try to find variants of this example in lower dimensions by considering heterotic string theory on tori. However in this case there is a T-duality transformation that maps the original charge vector \vec{Q} to momentum and winding along a compact circle. The small black hole describing this state could have an underlying AdS_3 factor that combines the AdS_2 component of the near horizon geometry, and the circle along which the string carries momentum. In this case the equality of the macroscopic and microscopic entropy would follow from the non-renormalization of the central charge of the boundary CFT.¹⁶

We can however find extremal small black holes without AdS_3 near horizon geometry by considering heterotic string theory compactified on manifolds without an S^1 factor, e.g. K3 or Calabi-Yau three fold. Let us for definiteness consider heterotic string theory compactified on a Calabi-Yau 3-fold and consider an elementary heterotic string carrying left-moving charges and right-moving oscillator excitations satisfying level matching condition. The statistical entropy of the system is again given by $2\pi\sqrt{Q^2}$. Again we expect the system to be described by a small black hole with AdS_2 near horizon geometry but no underlying AdS_3 . Our arguments will imply that the macroscopic entropy of this black hole will match the statistical entropy.

In this case in fact we can give an argument showing that for large Q^2 the macroscopic entropy is also given by $2\pi\sqrt{Q^2}$, thereby verifying our conjecture. The result is based on a

¹⁶The central charges c_L and c_R of the boundary CFT, related to the appropriate gauge and gravitational Chern-Simons terms of the bulk theory, has been carried out only for five-dimensional black strings i.e. four dimensional black holes. It would be interesting to do this computation for higher dimensional small black holes and verify that the central charges c_L and c_R of the boundary theory agree with the central charges of the fundamental heterotic string world-sheet theory.

universality argument similar to the one given in [65] for BPS black holes. We begin with the observation that since the string coupling square near the horizon is of order $1/\sqrt{\vec{Q}^2}$, we can carry out our analysis using tree level effective action. On the other hand the part of the tree level effective action that is relevant for our computation is the one that involves the metric, the Maxwell field and the dilaton and is independent of the manifold on which the theory is compactified. Thus we can replace the Calabi-Yau three fold by T^6 without changing the result for the microscopic entropy. In this case however the black hole under consideration can be rotated by T-duality to the one that carries only momentum and winding along a circle. For this system there is an AdS_3 factor in the near horizon geometry and we can calculate the entropy using the Kraus-Larsen argument to be $2\pi\sqrt{\vec{Q}^2}$. Thus the small black hole in heterotic string theory on T^6 must also have entropy $2\pi\sqrt{Q^2}$ in agreement with the microscopic entropy.

The final example we will consider is that of the entropy of possible small black holes describing fundamental type II strings. Let us consider an appropriate compactification of type IIA or IIB string theory down to four non-compact dimensions where the compactification breaks all the space-time supersymmetries in the left-moving sector of the world-sheet and preserves at least $\mathcal{N} = 2$ supersymmetry in the right-moving sector. We will also assume that the compact space contains an S^1 factor. Examples of such compactifications can be found in [66]. We now consider an elementary type II string in this theory, wound w times along S^1 and carrying momentum n along S^1 . For $nw > 0$ we can get extremal BPS states by keeping all the right-moving oscillators in their ground state and exciting the left-moving oscillators to level nw . On the other hand for $nw < 0$ we can get extremal non-BPS states by keeping all the left-moving oscillators in their ground state and exciting the right-moving oscillators to level $|nw|$. For large $|nw|$ the statistical entropy, computed from the degeneracy of states, is given in both cases by

$$S_{\text{stat}} = 2\sqrt{2}\pi\sqrt{|nw|}. \tag{7.17}$$

We would naively expect that in analogy with the heterotic example, the gravitational description of this system will be a small black hole. In fact a scaling argument along the line of [41, 42, 65] shows that the string coupling square at the horizon goes as $1/\sqrt{|nw|}$ so that to leading order we can consider only the tree level effective action, and the contribution to the entropy at tree level, if non-zero, must be proportional to $\sqrt{|nw|}$. Furthermore, the part of the tree level effective action relevant for computing the entropy is invariant under the world-sheet parity transformation to all orders in the α' expansion since it does not know about the left-right asymmetry introduced by the compactification. As a result the constant of proportionality in the expression for the entropy must be the same for both BPS and the non-BPS black holes. Thus if the agreement between the statistical and macroscopic entropy holds for extremal BPS black holes, it must also hold for extremal non-BPS black holes.

If as in heterotic string theory we proceed with the assumption that the AdS_2 factor of the near horizon geometry combines with the S^1 factor to give a locally AdS_3 space, we run into inconsistent results. Essentially the coefficients of the relevant Chern-Simons

terms vanish in the tree level type II effective action, and as result c_L and c_R appearing in (7.7) would vanish. Thus we will get vanishing answer for the entropy in disagreement with the statistical entropy both for the BPS and the non-BPS systems. Put another way, in this case the entropy function has no non-trivial extremum where the condition (7.4) is satisfied.

The only possible way out seems to be that the entropy function now has a different extremum at which the condition (7.4) is not satisfied. As a result the near horizon geometry does not have a locally AdS_3 factor. If there is indeed such a non-trivial extremum, then by the general scaling argument the macroscopic entropy, represented by the value of the entropy function at this extremum, will be proportional to $\sqrt{|nw|}$. At present we do not know if the entropy function has such an extremum, and even if has such an extremum, what would be the precise coefficient appearing in front of $\sqrt{|nw|}$. All we can say is that if this procedure leads to a macroscopic entropy that agrees with the statistical entropy for BPS black holes, then similar agreement would also be present for extremal non-BPS black holes.

7.3 Black holes from black strings

Typically in cases where the near horizon geometry is described by a locally AdS_3 space, the microscopic description of the black hole involves a string-like object wrapped along an internal circle S^1 , where the string itself may be the result of wrapping some brane configuration on an internal manifold. The charge n conjugate to the electric flux through AdS_2 has the interpretation of momentum carried by the string along the internal circle. If in the infrared the world-sheet theory of the string flows to a conformal field theory with central charges (C_L, C_R) then for large $|n|$ the statistical entropy of extremal states in this CFT, carrying only left-moving or only right-moving excitations, is given by the Cardy formula:

$$\begin{aligned}
 S_{\text{stat}} &= 2\pi\sqrt{\frac{C_L n}{6}} \quad \text{for } n > 0, \\
 &= 2\pi\sqrt{\frac{C_R |n|}{6}} \quad \text{for } n < 0.
 \end{aligned}
 \tag{7.18}$$

Note that in the $(0, 4)$ SCFT the states carrying left-moving momentum ($n > 0$) are BPS but states with right-moving momentum ($n < 0$) are non-BPS.

Let us now consider two possibilities. Let λ denote the parameter that controls the strength of the interaction in the world-sheet theory of the string. If λ is a marginal deformation of the CFT then we can vary it continuously. (This should correspond to the case where in the black hole description the attractor equations leave λ undetermined.) Since in a two dimensional CFT the central charges do not change under a marginal deformation, we can compute them for small λ by ignoring all interactions. This will then also give their values at large λ where the black hole description is good. The second possibility is that λ is not a marginal deformation and that in the infrared it gets fixed to a strong coupling value so that the dual black hole description has a horizon geometry with small curvature. In this case however we cannot calculate the central charges in

the microscopic theory directly. But for the special situation when the two dimensional boundary CFT has $(0,4)$ super conformal symmetry this is possible. The key point is that the $(0,4)$ world-sheet supersymmetry acting on the right-moving modes has $SU(2)_R$ R-symmetry. Furthermore supersymmetry relates the anomaly in the $SU(2)_R$ R-symmetry to the central charge C_R . Thus the calculation of C_R at strong coupling can be related to the calculation of the $SU(2)_R$ anomaly at strong coupling. The latter on the other hand is not renormalized beyond one loop. Thus knowing the perturbative answer for C_R we can calculate C_R and hence the statistical entropy at strong coupling for non-supersymmetric extremal black holes. Moreover, the quantity $C_L - C_R$ is related to the gravitational anomaly of the world-sheet theory of the string. Hence this is also not renormalized as we go from weak to strong coupling by the 't Hooft anomaly matching requirement. Thus we can also calculate C_L , and hence the statistical entropy of supersymmetric extremal black holes at strong coupling.

The non-renormalization of C_L and C_R as we go from weak to strong coupling regime shows that the statistical entropy of these systems do not change as we go from the weak to the strong coupling regime. As a result we should be able to compare the statistical entropy computed in the weakly coupled regime to the black hole entropy computed in the strong coupling regime. These arguments provide an alternate explanation of why the entropy of an extremal non-BPS black hole, calculated at strong coupling, should agree with the statistical entropy computed at weak coupling. It also provides an alternative explanation of why for large n the number of BPS states do not change as we go from weak coupling to the strong coupling region. However this argument is less powerful than the one based on supersymmetry, since this holds only in the limit of large $|n|$ when the statistical entropy is determined by the central charge alone.

One cannot fail to notice the similarity between (7.7) and (7.18). As already noted, using anomaly inflow one can relate the quantities c_L and c_R appearing in (7.7) to the left- and right-moving trace anomalies in the CFT living on the boundary of AdS_3 . If one further assumes AdS/CFT correspondence [52] then the CFT living on the boundary of AdS_3 is the same as the CFT describing the dynamics of the microscopic theory. This allows us to identify c_L and c_R with C_L and C_R respectively, and makes the equality of black hole entropy and statistical entropy manifest.

The arguments presented above require that the microscopic description be based on the dynamics of a string-like object, which may not always be the case. This is what happens for the example described in § 7.2 involving black holes in M-theory compactified on a non-elliptically fibered Calabi-Yau three fold. Moreover, even when there is an underlying string, determination of the central charges alone is not sufficient if one wishes to go beyond the leading asymptotics given by the Cardy entropy. This is the counterpart of the macroscopic result that the AdS_3 description is useful in the limit of large $|n|$, but fails when all the charges are of the same order. The arguments based on AdS_2 near horizon geometry continues to hold in such cases. Thus for these examples our conjecture makes nontrivial predictions about the relation between weak coupling statistical entropy and ‘strong’ coupling black hole entropy which would be interesting to verify.

8. Rotating black holes

Extremal spinning black holes also display attractor behavior which can be understood from the existence of the underlying entropy function [67]. Thus we expect the agreement between microscopic and macroscopic entropy to hold even in the case of spinning black holes.

An example of this may be constructed as follows. Let us consider the D1-D5 system with momentum considered in § 4.1 and add equal angular momentum in the two planes transverse to the D5-brane. Since for negative n the system was not supersymmetric to begin with, it will be non-BPS even after we add angular momentum. The entropy of this black hole can be computed directly, but can also be related to the entropy of a four dimensional black hole [68, 69] by taking the space transverse to the brane to be Taub-NUT space. This has the effect of compactifying an additional dimension (say x^4) with the angular momentum interpreted as the momentum along x^4 . Since the presence of the Taub-NUT space does not affect the structure of the black hole horizon in the limit where the size of the Taub-NUT space is large, in this limit the black hole entropy will be given by that of the rotating five dimensional black hole. On the other hand if we take the Taub-NUT space to be of small size then it is more appropriate to regard the black hole as a four dimensional black hole and the entropy will be given by the entropy of a four dimensional black hole carrying momentum along the x^4 direction. This is precisely the system described in § 4.2. Since the entropy cannot depend on the size of the Taub-NUT which, being the asymptotic radius of x^4 , is one of the moduli, we see that the entropies of the five and four dimensional black holes must be identical. On the other hand the microscopic counting of the four and the five dimensional systems are almost identical, with the four dimensional system receiving some additional contribution from the dynamics of the Taub-NUT space and the motion of the D1-D5 system in the Taub-NUT background [64]. However these contributions are subleading and do not affect the leading entropy in the limit of large charges. Thus the black hole entropy of the five dimensional rotating black hole agrees with the statistical entropy of the same system as a consequence of the corresponding agreement for the four dimensional non-rotating system discussed in § 4.2.

A closely related example is as follows¹⁷[19]. Let us consider type IIA string theory compactified on \mathcal{M} , where \mathcal{M} can be $K3 \times T^2$, T^6 or a Calabi-Yau 3-fold, and take a system of q_0 D0-branes and one D6 brane in this theory. Using the duality between type IIA string theory and M-theory on S^1_M , this configuration lifts to M-theory on Taub-NUT space $\times \mathcal{M}$ with q_0 units of momentum flowing along the asymptotic circle S^1_M of the Taub-NUT space. If the asymptotic radius of the M -theory circle S^1_M is big, then the center of Taub-NUT space is approximately flat 4 + 1 dimensional space-time $R^{4,1}$. The D0-brane charge q_0 now can be interpreted as equal angular momentum along a pair of orthogonal planes in R^4 and we get a neutral extremal rotating black hole sitting in the center of this approximately flat 4 + 1 dimensional space-time. If we denote the rotation group $SO(4)$ of $R^{4,1}$ by $SU(2)_L \times SU(2)_R$, then the black hole has q_0 units of angular momentum lying in $SU(2)_L$.

¹⁷A discussion for the attractor mechanism being the basis of the agreement between the microscopic and macroscopic entropy in this example also appears in the forth-coming paper [70].

This system breaks all supersymmetries. The Bekenstein-Hawking entropy of the black hole can be easily computed and is given by,

$$S_{\text{BH}} = \pi|q_0|. \tag{8.1}$$

We can in fact consider a more general class of black holes which also carry angular momentum J_R associated with $SU(2)_R$. From the five dimensional viewpoint this would correspond to having unequal angular momentum along the two orthogonal planes of R^4 . From the four dimensional viewpoint this describes an extremal charged rotating black hole. The entropy of the corresponding black hole can also be computed easily and yields the answer

$$S_{\text{BH}} = 2\pi\sqrt{\frac{(q_0)^2}{4} - J_R^2}. \tag{8.2}$$

For $\mathcal{M} = T^6$ the microscopic entropy of this system was computed in [19] by studying the dynamics of the D0-D6 system and yields the answer

$$S_{\text{stat}} = 2\pi\sqrt{\frac{(q_0)^2}{4} - J_R^2}, \tag{8.3}$$

in agreement with the Bekenstein-Hawking entropy.¹⁸ Thus this provides an example where the macroscopic entropy of an extremal non-BPS black hole calculated at ‘strong’ coupling agrees with the statistical entropy of the system calculated at weak coupling.

In this system, the initial configuration, when interpreted as a rotating black hole solution in five dimensions, does not have an obvious AdS_3 factor. However interpreted as a four dimensional black hole this system is dual to heterotic string theory on T^6 for $\mathcal{M} = K3 \times T^2$ and type IIA string theory on T^6 for $\mathcal{M} = T^6$. Let us for definiteness concentrate on the case $\mathcal{M} = K3 \times T^2$; the case for $\mathcal{M} = T^6$ may be analyzed in a similar manner. If we set $J_R = 0$ then black hole solution describes a non-rotating black hole in four dimensions carrying some electric and magnetic charges (Q, P) with $Q^2 = P^2 = 0$, $Q \cdot P = q_0^2$. Since in the supergravity approximation the entropy is a function of the duality invariant combination $D \equiv [P^2 Q^2 - (P \cdot Q)^2]/4$, we can calculate the entropy by choosing a different representative with the same value of D that has an AdS_3 factor in its near horizon geometry. For example one can map this system to the familiar D1-D5-KK5- \bar{P} system of type I theory discussed in § 4.2. The entropy of this system is given by

$$S_{\text{BH}} = 2\pi\sqrt{-D} = \pi|q_0| \tag{8.4}$$

in agreement with (8.1). This allows us to use an AdS_3 based argument along the line of § 7.1 for explaining the agreement between the statistical and black hole entropy.

Note however that if we begin with a rotating extremal black hole configuration in M-theory on $K3 \times T^2 \times \text{Taub-NUT}$ where $K3$, T^2 and Taub-NUT have sizes large compared to the 11-dimensional Planck scale, and the angular momentum is large so that the horizon size is large compared to the Planck scale, then the original description in terms of M-theory

¹⁸We expect that a similar computation can be done at least for $\mathcal{M} = K3 \times T^2$.

is a weakly coupled description. A duality transformation that takes this to a system with an AdS_3 factor in the near horizon geometry must map it to a region of the moduli space where some degrees of freedom in the final description are strongly coupled since we cannot have two different weakly coupled descriptions of the same background. This could break the $SO(3,1)$ symmetry of Euclidean AdS_3 strongly by the various mechanisms discussed in § 7.1. In order to get an AdS_3 geometry with weakly coupled degrees of freedom, we must begin at a corner of the moduli space where the original description in terms of M-theory is strongly coupled. We then need to invoke the attractor mechanism to argue that the entropy of the system does not change as we move from the weakly coupled region to the strongly coupled region. The implicit use of attractor mechanism can also be seen from that fact that in order to argue that the entropy is a function only of $P^2Q^2 - (P \cdot Q)^2$ without doing explicit calculation, we need to assume that it does not depend on the asymptotic moduli. Otherwise we could construct more general duality invariant combinations of moduli and charges on which the entropy could depend.

Let us now consider the effect of switching on J_R in the original D0-D6 system. From the point of view of a (3+1) dimensional theory this corresponds to imparting an angular momentum on the system. The effect of this is to change (8.4) to [71–73, 67] (see eqs.(5.104), (5.105) of [67])

$$\begin{aligned} S_{\text{BH}} &= 2\pi\sqrt{D + J_R^2} \quad \text{for } D + J_R^2 > 0, \\ &= 2\pi\sqrt{-D - J_R^2} \quad \text{for } D + J_R^2 < 0. \end{aligned} \tag{8.5}$$

The case $D + J_R^2 < 0$ corresponds to the branch of the rotating D0-D6 black hole which has no ergo-sphere, while the case $D + J_R^2 > 0$ corresponds to the branch with an ergo-sphere [71, 72, 67]. Substituting $D = -(q_0)^2$ in the second equation of (8.5) we recover eq. (8.2).

Let us now return the case where \mathcal{M} is a Calabi-Yau 3-fold. First suppose \mathcal{M} is elliptically fibered with base B . Then by the usual M-theory - F-theory duality we can relate this to IIB on $B \times S^1$. After performing the complicated set of duality transformations described earlier we can arrive at a configuration where the near horizon geometry has an AdS_3 factor, and the black hole entropy can be calculated using eqs. (7.7) as usual. However if the manifold \mathcal{M} is not elliptically fibered then there is no obvious way at least to associate an AdS_3 space with the compactification, and therefore we cannot apply eq. (7.7) to compute the entropy. However a discussion analogous to that of [67] will apply for the five dimensional rotating black hole, showing that the attractor mechanism does work in this case as well. And thus the arguments presented in this paper will provide a prediction for the microscopic counting of states.

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References

- [1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [[hep-th/9601029](#)].
- [2] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ extremal black holes*, *Phys. Rev. D* **52** (1995) 5412 [[hep-th/9508072](#)].
- [3] A. Strominger, *Macroscopic entropy of $n = 2$ extremal black holes*, *Phys. Lett. B* **383** (1996) 39 [[hep-th/9602111](#)].
- [4] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev. D* **54** (1996) 1514 [[hep-th/9602136](#)].
- [5] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett. B* **451** (1999) 309 [[hep-th/9812082](#)].
- [6] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Deviations from the area law for supersymmetric black holes*, *Fortschr. Phys.* **48** (2000) 49 [[hep-th/9904005](#)].
- [7] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, *Nucl. Phys. B* **567** (2000) 87 [[hep-th/9906094](#)].
- [8] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $N = 2$ supergravity with R^2 interactions*, *JHEP* **12** (2000) 019 [[hep-th/0009234](#)].
- [9] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev. D* **70** (2004) 106007 [[hep-th/0405146](#)].
- [10] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301 [[hep-th/0409148](#)].
- [11] A. Dabholkar, R. Kallosh and A. Maloney, *A stringy cloak for a classical singularity*, *JHEP* **12** (2004) 059 [[hep-th/0410076](#)].
- [12] A. Sen, *How does a fundamental string stretch its horizon?*, *JHEP* **05** (2005) 059 [[hep-th/0411255](#)].
- [13] A. Sen, *Black holes, elementary strings and holomorphic anomaly*, *JHEP* **07** (2005) 063 [[hep-th/0502126](#)].
- [14] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Exact and asymptotic degeneracies of small black holes*, *JHEP* **08** (2005) 021 [[hep-th/0502157](#)].
- [15] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Precision counting of small black holes*, *JHEP* **10** (2005) 096 [[hep-th/0507014](#)].
- [16] D.M. Kaplan, D.A. Lowe, J.M. Maldacena and A. Strominger, *Microscopic entropy of $N = 2$ extremal black holes*, *Phys. Rev. D* **55** (1997) 4898 [[hep-th/9609204](#)].
- [17] G.T. Horowitz, D.A. Lowe and J.M. Maldacena, *Statistical entropy of nonextremal four-dimensional black holes and u-duality*, *Phys. Rev. Lett.* **77** (1996) 430 [[hep-th/9603195](#)].

- [18] A. Dabholkar, *Microstates of non-supersymmetric black holes*, *Phys. Lett. B* **402** (1997) 53 [[hep-th/9702050](#)].
- [19] R. Emparan and G.T. Horowitz, *Microstates of a neutral black hole in M-theory*, *Phys. Rev. Lett.* **97** (2006) 141601 [[hep-th/0607023](#)].
- [20] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys. B* **500** (1997) 75 [[hep-th/9702103](#)].
- [21] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [[hep-th/0506177](#)].
- [22] K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, *Non-supersymmetric attractors*, *Phys. Rev. D* **72** (2005) 124021 [[hep-th/0507096](#)].
- [23] R. Kallosh, *New attractors*, *JHEP* **12** (2005) 022 [[hep-th/0510024](#)].
- [24] P.K. Tripathy and S.P. Trivedi, *Non-supersymmetric attractors in string theory*, *JHEP* **03** (2006) 022 [[hep-th/0511117](#)].
- [25] P. Kaura and A. Misra, *On the existence of non-supersymmetric black hole attractors for two-parameter Calabi-Yau's and attractor equations*, [hep-th/0607132](#).
- [26] A. Giriyavets, *New attractors and area codes*, *JHEP* **03** (2006) 020 [[hep-th/0511215](#)].
- [27] K. Goldstein, R.P. Jena, G. Mandal and S.P. Trivedi, *A C-function for non-supersymmetric attractors*, *JHEP* **02** (2006) 053 [[hep-th/0512138](#)].
- [28] R. Kallosh, N. Sivanandam and M. Soroush, *The non-BPS black hole attractor equation*, *JHEP* **03** (2006) 060 [[hep-th/0602005](#)].
- [29] R. Kallosh, *From BPS to non-BPS black holes canonically*, [hep-th/0603003](#).
- [30] R.R. Khuri and T. Ortin, *A non-supersymmetric dyonic extreme Reissner-Nordstrom black hole*, *Phys. Lett. B* **373** (1996) 56 [[hep-th/9512178](#)].
- [31] T. Ortin, *Extremality versus supersymmetry in stringy black holes*, *Phys. Lett. B* **422** (1998) 93 [[hep-th/9612142](#)].
- [32] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008 [[hep-th/0508042](#)].
- [33] R.M. Wald, *Black hole entropy in the Noether charge*, *Phys. Rev. D* **48** (1993) 3427 [[gr-qc/9307038](#)].
- [34] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846 [[gr-qc/9403028](#)].
- [35] T. Jacobson, G. Kang and R.C. Myers, *On black hole entropy*, *Phys. Rev. D* **49** (1994) 6587 [[gr-qc/9312023](#)].
- [36] T. Jacobson, G. Kang and R.C. Myers, *Black hole entropy in higher curvature gravity*, [gr-qc/9502009](#).
- [37] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, *JHEP* **01** (2006) 022 [[hep-th/0508218](#)].
- [38] P. Kraus and F. Larsen, *Microscopic black hole entropy in theories with higher derivatives*, *JHEP* **09** (2005) 034 [[hep-th/0506176](#)].
- [39] P. Kraus, *Lectures on black holes and the AdS_3/CTF_2 correspondence*, [hep-th/0609074](#).

- [40] E. Witten and D.I. Olive, *Supersymmetry algebras that include topological charges*, *Phys. Lett.* **B 78** (1978) 97.
- [41] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett.* **A 10** (1995) 2081 [[hep-th/9504147](#)].
- [42] A.W. Peet, *Entropy and supersymmetry of D-dimensional extremal electric black holes versus string states*, *Nucl. Phys.* **B 456** (1995) 732 [[hep-th/9506200](#)].
- [43] A. Sen, *Stretching the horizon of a higher dimensional small black hole*, *JHEP* **07** (2005) 073 [[hep-th/0505122](#)].
- [44] C. Vafa, *Black holes and Calabi-Yau threefolds*, *Adv. Theor. Math. Phys.* **2** (1998) 207 [[hep-th/9711067](#)].
- [45] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Area law corrections from state counting and supergravity*, *Class. and Quant. Grav.* **17** (2000) 1007 [[hep-th/9910179](#)].
- [46] A. Dabholkar and J.A. Harvey, *Nonrenormalization of the superstring tension*, *Phys. Rev. Lett.* **63** (1989) 478.
- [47] A. Dabholkar, *Ten-dimensional heterotic string as a soliton*, *Phys. Lett.* **B 357** (1995) 307 [[hep-th/9506160](#)].
- [48] C.M. Hull, *String-string duality in ten-dimensions*, *Phys. Lett.* **B 357** (1995) 545 [[hep-th/9506194](#)].
- [49] J. Polchinski and E. Witten, *Evidence for heterotic - type-I string duality*, *Nucl. Phys.* **B 460** (1996) 525 [[hep-th/9510169](#)].
- [50] A. Strominger, *Open p-branes*, *Phys. Lett.* **B 383** (1996) 44 [[hep-th/9512059](#)].
- [51] J.M. Maldacena and A. Strominger, *Statistical entropy of four-dimensional extremal black holes*, *Phys. Rev. Lett.* **77** (1996) 428 [[hep-th/9603060](#)].
- [52] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [[hep-th/9711053](#)].
- [53] B. Sahoo and A. Sen, *Btz black hole with Chern-Simons and higher derivative terms*, *JHEP* **07** (2006) 008 [[hep-th/0601228](#)].
- [54] A. Ghodsi, *R^4 corrections to D1D5P black hole entropy from entropy function formalism*, *Phys. Rev.* **D 74** (2006) 124026 [[hep-th/0604106](#)].
- [55] B. Sahoo and A. Sen, *α' corrections to extremal dyonic black holes in heterotic string theory*, [hep-th/0608182](#).
- [56] J. Preskill, P. Schwarz, A.D. Shapere, S. Trivedi and F. Wilczek, *Limitations on the statistical description of black holes*, *Mod. Phys. Lett.* **A 6** (1991) 2353.
- [57] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)].
- [58] A. Strominger, *Black hole entropy from near-horizon microstates*, *JHEP* **02** (1998) 009 [[hep-th/9712251](#)].
- [59] S.N. Solodukhin, *Holography with gravitational Chern-Simons*, *Phys. Rev.* **D 74** (2006) 024015 [[hep-th/0509148](#)].

- [60] H. Saida and J. Soda, *Statistical entropy of BTZ black hole in higher curvature gravity*, *Phys. Lett. B* **471** (2000) 358 [[gr-qc/9909061](#)].
- [61] P. Kraus and F. Larsen, *Partition functions and elliptic genera from supergravity*, [hep-th/0607138](#).
- [62] J.A. Harvey and G.W. Moore, *Fivebrane instantons and R^2 couplings in $N = 4$ string theory*, *Phys. Rev. D* **57** (1998) 2323 [[hep-th/9610237](#)].
- [63] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [[hep-th/0412287](#)].
- [64] J.R. David and A. Sen, *CHL dyons and statistical entropy function from D1-D5 system*, *JHEP* **11** (2006) 072 [[hep-th/0605210](#)].
- [65] A. Sen, *Black holes and elementary string states in $N = 2$ supersymmetric string theories*, *JHEP* **02** (1998) 011 [[hep-th/9712150](#)].
- [66] A. Sen and C. Vafa, *Dual pairs of type-II string compactification*, *Nucl. Phys. B* **455** (1995) 165 [[hep-th/9508064](#)].
- [67] D. Astefanesei, K. Goldstein, R.P. Jena, A. Sen and S.P. Trivedi, *Rotating attractors*, *JHEP* **10** (2006) 058 [[hep-th/0606244](#)].
- [68] D. Gaiotto, A. Strominger and X. Yin, *New connections between 4D and 5D black holes*, *JHEP* **02** (2006) 024 [[hep-th/0503217](#)].
- [69] D. Gaiotto, A. Strominger and X. Yin, *5D black rings and 4D black holes*, *JHEP* **02** (2006) 023 [[hep-th/0504126](#)].
- [70] D. Astefanesei, K. Goldstein, and S. Mahapatra, *Moduli and (un)attractor black hole thermodynamics*, to appear.
- [71] D. Rasheed, *The rotating dyonic black holes of Kaluza-Klein theory*, *Nucl. Phys. B* **454** (1995) 379 [[hep-th/9505038](#)].
- [72] F. Larsen, *Rotating Kaluza-Klein black holes*, *Nucl. Phys. B* **575** (2000) 211 [[hep-th/9909102](#)].
- [73] M. Cvetič and D. Youm, *Entropy of non-extreme charged rotating black holes in string theory*, *Phys. Rev. D* **54** (1996) 2612 [[hep-th/9603147](#)].